22117203

International Baccalaureate
Baccalauréat International
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Wahbeh
Key

MATHEMATICS
HIGHER LEVEL
PAPER 1

Candidate session number

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Wednesday 4 May 2011 (afternoon)

2 hours

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 7]*

Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.4$.

(a) Find the value of $\mathrm{P}(A \cup B)$ when

(i) A and B are mutually exclusive;

(ii) A and B are independent. *[4 marks]*

(b) Given that $\mathrm{P}(A \cup B) = 0.6$, find $\mathrm{P}(A \mid B)$. *[3 marks]*

ai $P(A \cup B) = P(A) + P(B) = .3 + .4 = \boxed{.7}$

ii $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= P(A) + P(B) - P(A) \cdot P(B)$
$= .3 + .4 - .3(.4)$
$= .7 - .12 = \boxed{.58}$

b. $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
$= .3 + .4 - .6 = .1$

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{.1}{.4} = \boxed{\dfrac{1}{4}}$

2. *[Maximum mark: 4]*

Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

$z = (2-i)(z+2)$

$z = 2z + 4 - iz - 2i$

$iz - 1z = 4 - 2i$

$z(i-1) = 4 - 2i$

$z = \left(\frac{4-2i}{i-1}\right) \cdot \left(\frac{i+1}{i+1}\right)$

$= \frac{4i + 4 - 2i^2 - 2i}{i^2 + i - i - 1} = \frac{2i + 4 - 2(-1)}{-1-1}$

$z = \frac{2i+6}{-2}$

$\boxed{z = -3 - i}$

3. *[Maximum mark: 7]*

A geometric sequence u_1, u_2, u_3, ... has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$.

(a) Find the common ratio of the geometric sequence. *[2 marks]*

An arithmetic sequence v_1, v_2, v_3, ... is such that $v_2 = u_2$ and $v_4 = u_4$.

(b) Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$. *[5 marks]*

a. $S = \frac{u_1}{1-r} \Rightarrow \frac{81}{2} = \frac{27}{1-r} \Rightarrow 1-r = \frac{2}{\cancel{81}_3} \cdot \cancel{27}^1$

$1-r = \frac{2}{3} \Rightarrow \boxed{r = \frac{1}{3}}$

b. $u_n = u_1 r^{n-1} \Rightarrow u_2 = 27\left(\frac{1}{3}\right)^1 = 9 = v_2$

$u_4 = 27\left(\frac{1}{3}\right)^3 = 1 = v_4$

$u_n = u_1 + (n-1)d \Rightarrow u_2 = u_1 + 1d = 9$

$u_4 = u_1 + 3d = 1$

$2d = -8 \Rightarrow d = -4$

$u_2 \Rightarrow u_1 - 4 = 9$

$\boxed{u_1 = 13}$

$S_N = \frac{N}{2}(2u_1 + (N-1)d)$

$= \frac{N}{2}(2(13) + (N-1)(-4)) > 0$

$= \frac{N}{2}(26 - 4N + 4) > 0$

$= \frac{N}{2}(30 - 4N) > 0$

$\frac{N}{2} > 0$ or $30 - 4N > 0$

$N > 0$ $\quad 4N < 30$

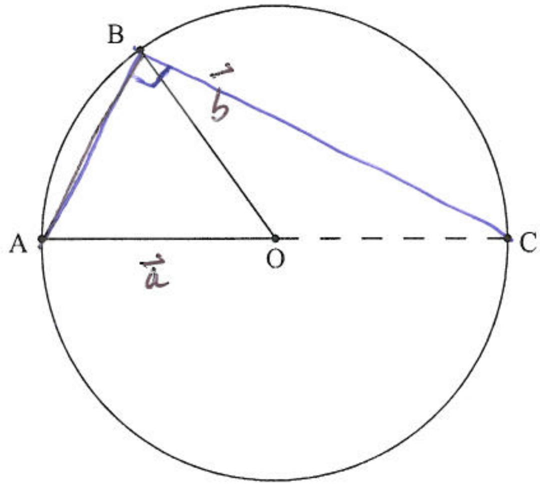
$\to N < \frac{30}{4}$

$N < 7.5$

$\boxed{N = 7}$

4. *[Maximum mark: 5]*

The diagram below shows a circle with centre O. The points A, B, C lie on the circumference of the circle and [AC] is a diameter.



Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

(a) Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{CB}$ in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. *[2 marks]*

(b) Hence prove that angle $A\hat{B}C$ is a right angle. *[3 marks]*

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b - a$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$

$= \overrightarrow{OB} - (-\vec{a})$ where $\overrightarrow{OC} = -\vec{a}$

$= b + a$

b) $\overrightarrow{AB} \cdot \overrightarrow{CB}$ should equal to zero.

$\overrightarrow{AB} \cdot \overrightarrow{CB} = (b-a)\cdot(b+a)$

$= b^2 + ab - ab - a^2$

$= |b|^2 - |a|^2$

$= 0$ since $|b| = |a|$

Think: They are both radii

so $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CB}$ $\therefore$ $A\hat{B}C$ is a right angle.

5. *[Maximum mark: 5]*

(a) Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. *[2 marks]*

(b) Hence find the value of $\cot\frac{\pi}{8}$ in the form $a+b\sqrt{2}$, where $a, b \in \mathbb{Z}$. *[3 marks]*

a) $\sin 2\theta = 2\sin\theta\cos\theta$

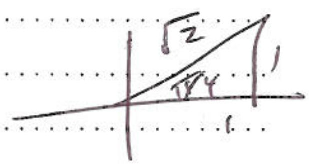
$\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore \frac{2\sin\theta\cos\theta}{1+(1-2\sin^2\theta)} = \frac{2\sin\theta\cos\theta}{2-2\sin^2\theta} = \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}(1-\sin^2\theta)}$

$= \frac{\sin\theta\cos\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\therefore \tan\theta = \tan\theta$ Q.E.D.

b. $\tan\frac{\pi}{8} = \frac{\sin 2\cdot\frac{\pi}{8}}{1+\cos 2\cdot\frac{\pi}{8}}$



$\cot\frac{\pi}{8} = \frac{1+\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$

$= \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{1}{\frac{\sqrt{2}}{2}} + \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= \frac{2}{\sqrt{2}} + 1$

$= \frac{\cancel{2}\sqrt{2}}{\cancel{2}} + 1$

$\boxed{1 + 1\sqrt{2}}$

6. *[Maximum mark: 5]*

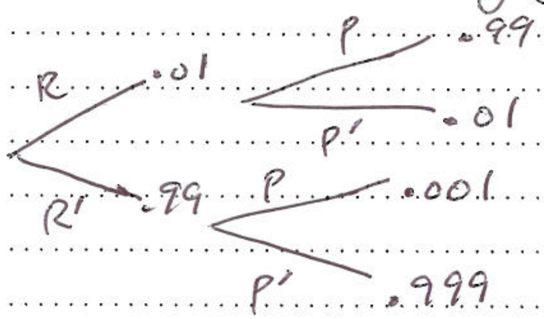
In a population of rabbits, 1 % are known to have a particular disease. A test is developed for the disease that gives a positive result for a rabbit that **does** have the disease in 99 % of cases. It is also known that the test gives a positive result for a rabbit that **does not** have the disease in 0.1 % of cases. A rabbit is chosen at random from the population.

(a) Find the probability that the rabbit tests positive for the disease. *[2 marks]*

(b) Given that the rabbit tests positive for the disease, show that the probability that the rabbit does not have the disease is less than 10 %. *[3 marks]*

Let R = rabbit with disease

P = rabbit testing positive for the disease.



a. $P(P) = P(R \cap P) + P(R' \cap P)$

$= .01(.99) + .99(.001)$

$= .0099 + .00099$

$$\begin{array}{r} .00099 \\ +.0099 \\ \hline .01089 \end{array}$$

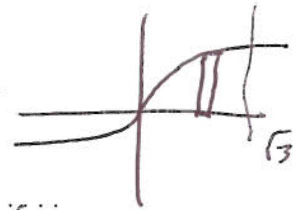
$\boxed{= .01089 \approx .0109}$

b. $P(R'|P) = \dfrac{P(R' \cap P)}{P(P)} = \dfrac{.00099}{.01089} < .1$

$\boxed{= .00099 < .001089 \quad \text{True}}$

7. *[Maximum mark: 6]*

Find the area enclosed by the curve $y = \arctan x$, the x-axis and the line $x = \sqrt{3}$.



$A = \int_0^{\sqrt{3}} \tan^{-1} x \, dx$　　by parts.

$u = \tan^{-1} x$, $dv = 1\,dx$

$du = \frac{1}{1+x^2}\,dx$, $v = x$

$\int u\,dv = uv - \int v\,du$

$= x\tan^{-1} x - \frac{1}{2}\int x \cdot \frac{1}{1+x^2}\,dx \ (2)$　let $m = 1+x^2$

$dm = 2x\,dx$

$-\frac{1}{2}\int \frac{dm}{m}$

$-\frac{1}{2}\ln m$

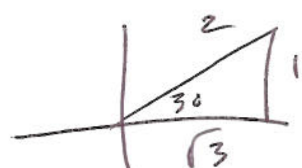
$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) \Big|_0^{\sqrt{3}}$

$= \left[\sqrt{3}\tan^{-1}\sqrt{3} - \frac{1}{2}\ln(1+\sqrt{3}^2)\right] - \left[0 - \cancel{\frac{1}{2}\ln 1}\right]$　0

$= \sqrt{3}\cdot\frac{\pi}{3} - \frac{1}{2}\ln 4$

$= \frac{\sqrt{3}\pi}{3} - \ln 4^{1/2}$

$\boxed{= \frac{\sqrt{3}\pi}{3} - \ln 2}$



$\tan\frac{\pi}{3} = \sqrt{3}$

8. *[Maximum mark: 6]*

Consider the functions given below.

$$f(x) = 2x + 3$$
$$g(x) = \frac{1}{x},\ x \neq 0$$

(a) (i) Find $(g \circ f)(x)$ and write down the domain of the function.

(ii) Find $(f \circ g)(x)$ and write down the domain of the function. *[2 marks]*

(b) Find the coordinates of the point where the graph of $y = f(x)$ and the graph of $y = (g^{-1} \circ f \circ g)(x)$ intersect. *[4 marks]*

a i. $(g \circ f)(x) = g(f(x)) = g(2x+3) = \frac{1}{2x+3}$, D: $x \neq -\frac{3}{2}$

ii. $(f \circ g)(x) = f(g(x)) = f(\frac{1}{x}) = 2 \cdot \frac{1}{x} + 3$, D: $x \neq 0$

b. $g^{-1}(f(g(x))) = g^{-1}(\frac{2}{x}+3)$, $g(x) = \frac{1}{x}$

$\therefore g^{-1}(\frac{2}{x}+\frac{3}{1}) = g^{-1}\left(\frac{2+3x}{x}\right) = \frac{1}{\frac{2+3x}{x}}$

$y = \frac{1}{x}$

$x = \frac{1}{y}$

$y^{-1} = \frac{1}{x} \therefore g^{-1}(x) = \frac{1}{x}$

$= \frac{x}{2+3x}$

Since $y = (g^{-1} \circ f \circ g)(x)$ and $y = 2x+3$

$\therefore \frac{x}{2+3x} = 2x+3$

$x = (2x+3)(2+3x)$

$0 = 4x + 6x^2 + 6 + 9x - x$

$0 = 6x^2 + 12x + 6$

$= 6(x^2 + 2x + 1)$

$= 6(x+1)(x+1)$

$\boxed{x = -1}$

so $y = 2(-1) + 3 = 1$

$\boxed{(-1, 1)}$

1015

9. *[Maximum mark: 7]*

Show that the points $(0, 0)$ and $\left(\sqrt{2\pi}, -\sqrt{2\pi}\right)$ on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent.

Take derivative of both sides

$e^{(x+y)} \cdot (1+y') = -\underbrace{\sin(xy)}_{0} \cdot (1y+xy')$

Let $x=0$, $y=0$

$e^{0}(1+y') = 0$

$1+y'=0 \Rightarrow \boxed{y'=-1}$

Let $x=\sqrt{2\pi}$, $y=-\sqrt{2\pi}$

$e^{\sqrt{2\pi}-\sqrt{2\pi}}(1+y') = -\cancelto{0}{\sin(\sqrt{2\pi}\sqrt{-2\pi})}\left(-\sqrt{2\pi}+\sqrt{2\pi}y'\right)$

Same gradient

$e^{0}(1+y')=0$

$1+y'=0 \Rightarrow \boxed{y'=-1}$

$y-y_1 = m(x-x_1)$

$y-0 = -1(x-0)$

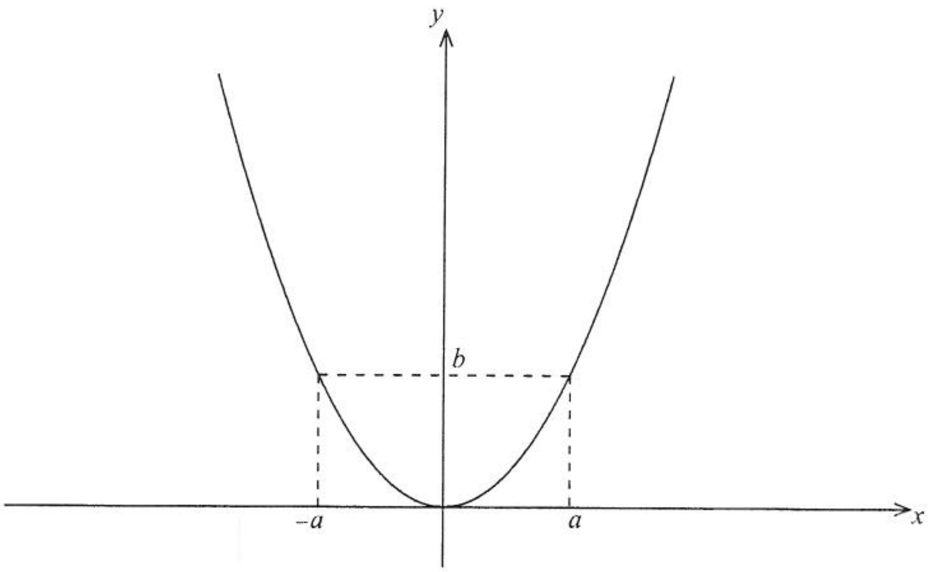
$y=-x$ is the line that both points lie on

is the common tangent.

1115

10. *[Maximum mark: 8]*

The diagram below shows the graph of the function $y = f(x)$, defined for all $x \in \mathbb{R}$, where $b > a > 0$.



Consider the function $g(x) = \dfrac{1}{f(x-a)-b}$.

(a) Find the largest possible domain of the function g. *[2 marks]*

When $X = 2a$, $g(2a) = \dfrac{1}{f(2a-a)-b} = \dfrac{1}{f(a)-b}$

$= \dfrac{1}{b-b} = \dfrac{1}{0}$ $\therefore$ $\boxed{X \neq 2a}$

When $X = 0$, $g(0) = \dfrac{1}{f(-a)-b} = \dfrac{1}{b-b} = \dfrac{1}{0}$ $\therefore$ $\boxed{X \neq 0}$

Domain of g is all R except $X \neq 2a, 0$

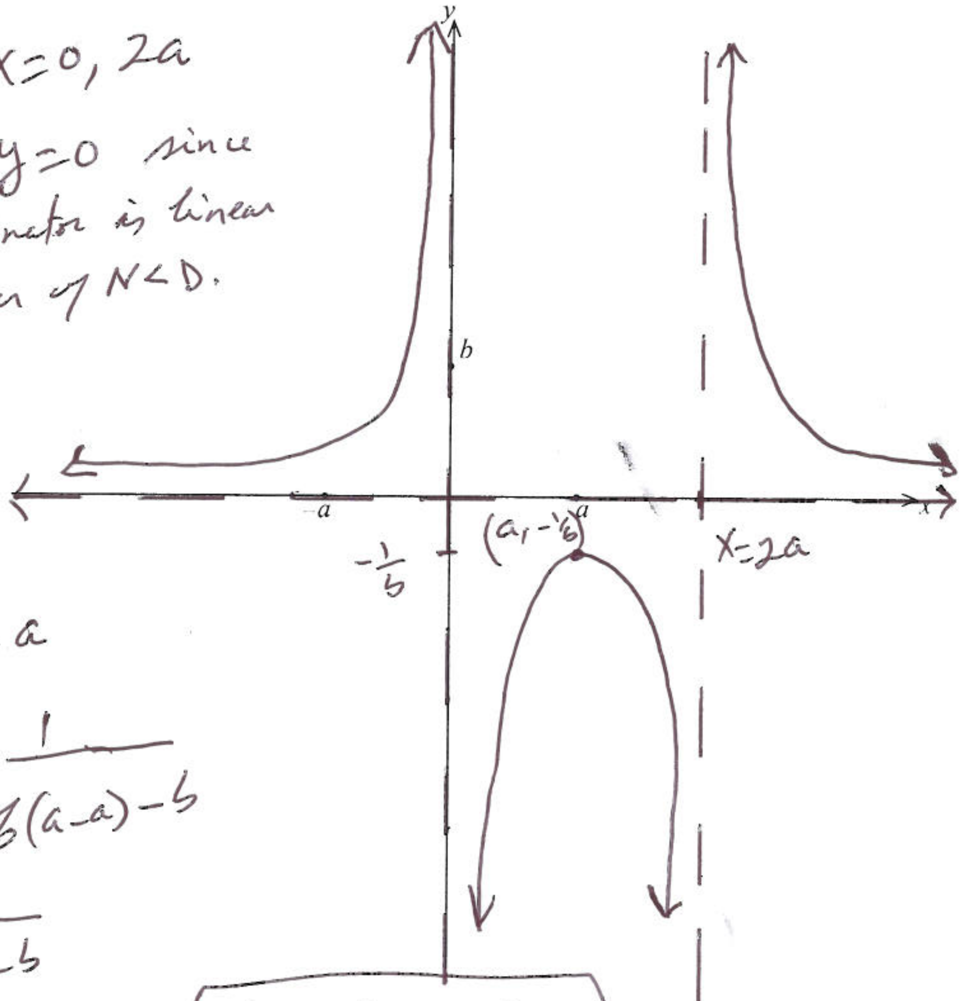
(This question continues on the following page)

(Question 10 continued)

(b) On the axes below, sketch the graph of $y = g(x)$. On the graph, indicate any asymptotes and local maxima or minima, and write down their equations and coordinates. *[6 marks]*

V.A. $x=0, 2a$

H.A. $y=0$ since Denominator is linear and power of $N < D$.



let $x=a$

$$g(a) = \frac{1}{b(a-a)-b}$$

$$= \frac{1}{b(0)-b}$$

$$= \frac{1}{0-b} = -\frac{1}{b} \quad \therefore \boxed{\left(a, -\tfrac{1}{b}\right) \text{ local Max}}$$

let $x = 3a$

$$g(3a) = \frac{1}{b(3a-a)-b} = \frac{1}{b(2a)-b} \approx \frac{1}{2b-b} = \frac{1}{b}$$ Truly some positive value.

$$g(-3a) = \frac{1}{b(-3a-a)-b} = \frac{1}{b(-4a)-b} \approx \frac{1}{4b-b} = \frac{1}{3b}$$ some + value.

g: $+$ | 0 | $-$ | $2a$ | $+$

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 19]*

The points $A(1, 2, 1)$, $B(-3, 1, 4)$, $C(5, -1, 2)$ and $D(5, 3, 7)$ are the vertices of a tetrahedron.

$\vec{AB} = B - A = \langle -4, -1, 3 \rangle$

$\vec{AC} = C - A = \langle 4, -3, 1 \rangle$

(a) Find the vectors $\vec{AB}$ and $\vec{AC}$. *[2 marks]*

(b) Find the Cartesian equation of the plane Π that contains the face ABC. *[4 marks]*

(c) Find the vector equation of the line that passes through D and is perpendicular to Π. Hence, or otherwise, calculate the shortest distance to D from Π. *[5 marks]*

(d) (i) Calculate the area of the triangle ABC.

(ii) Calculate the volume of the tetrahedron ABCD. *[4 marks]*

(e) Determine which of the vertices B or D is closer to its opposite face. *[4 marks]*

b) $\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -4 & -1 & 3 \\ 4 & -3 & 1 \end{vmatrix} = (-1+9)i - (-4-12)j + (12+4)k = 8i + 16j + 16k \Rightarrow n = \begin{pmatrix} 8 \\ 16 \\ 16 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$ax + by + cz + d = 0$ eq. of plane. when $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$1(1) + 2(2) + 2(1) + d = 0 \Rightarrow d = -7$

$\boxed{\therefore\ x + 2y + 2z = 7}$ $\boxed{\text{eq. of plane } \Pi}$

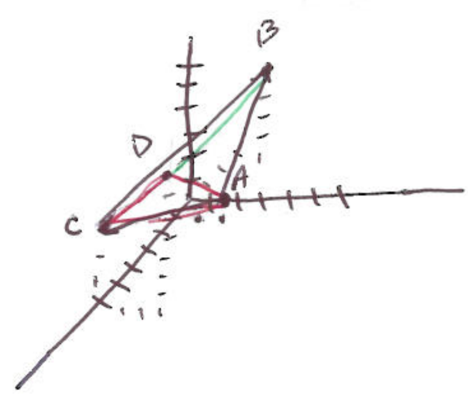
c) $r = a + \lambda b$

$r = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

shortest distance.

$D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$ absolute value

$= \dfrac{|1(5) + 2(3) + 2(7) - 7|}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{18}{3} = 6$



11 d i. $A = \frac{1}{2}\left|\vec{AB} \times \vec{BC}\right| = \frac{1}{2}\left|\begin{matrix}1\\2\\2\end{matrix}\right| = \frac{1}{2}\cdot\sqrt{8^2+16^2+16^2}$

$= \frac{1}{2}\sqrt{64+256+256} = \boxed{\frac{1}{2}\sqrt{576} \text{ accept.}} = \frac{1}{2}\cdot 24 = \boxed{12}$

ii Volume of tetrahedron $= \frac{1}{3}\cdot \text{area} \cdot \text{height}$ (See formula in IB packet)

$= \frac{1}{3}\cdot 12 \cdot 6$

$\boxed{= 24}$

e) $\vec{AD} = D - A = \langle 4, 1, 6 \rangle$

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 4 & -3 & 1 \\ 4 & 1 & 6 \end{vmatrix} = (-18-1)i - (24-4)j + (4+12)k$$

$$= -19i - 20j + 16k \Rightarrow n = \begin{pmatrix}-19\\-20\\16\end{pmatrix}$$

Eq. of plane: $ax + by + cz + d = 0$

$-19(1) - 20(2) + 16(1) + d = 0$

$-19 - 40 + 16 + d = 0 \Rightarrow d = 43$

$\therefore -19x - 20y + 16z - 43 = 0$ eq. of plane

$$\text{Distance} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|-3(-19) + 1(-20) + 4(16) + 43|}{\sqrt{19^2+20^2+16^2}}$$

$$= \frac{144}{\sqrt{298+400+256}} = \frac{144}{\sqrt{954}} \approx \frac{144}{31} \approx 5 \quad \text{point B}$$

since point D = 6 and B $\approx$ 5 $\therefore$ point B is closer to its opposite face.

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

12. *[Maximum mark: 19]*

Consider the function $f(x) = \frac{\ln x}{x}$, $0 < x < e^2$.

(a) (i) Solve the equation $f'(x) = 0$.

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f. *[5 marks]*

(b) Show that there is a point of inflexion on the graph and determine its coordinates. *[5 marks]*

(c) Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. *[3 marks]*

(d) Now consider the functions $g(x) = \frac{\ln|x|}{x}$ and $h(x) = \frac{\ln|x|}{|x|}$, where $0 < |x| < e^2$.

(i) Sketch the graph of $y = g(x)$.

(ii) Write down the range of g.

(iii) Find the values of x such that $h(x) > g(x)$. *[6 marks]*

13. *[Maximum mark: 22]*

(a) Write down the expansion of $(\cos\theta + i\sin\theta)^3$ in the form $a + ib$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. *[2 marks]*

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. *[3 marks]*

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. *[3 marks]*

(d) **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. *[6 marks]*

(e) By considering the solutions of the equation $\cos 5\theta = 0$, show that $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$. *[8 marks]*

12. $f(x) = \frac{\ln x}{x}$

a i $f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} = 0$

$1 - \ln x = 0$

$e^{\ln x} = e^{1}$

$\boxed{x = e}$

ii $f'(1) = \frac{1-0}{1^2} = 1$ which is +.

$\longleftarrow\!\!\!\!\!\!\;\;\;\;\;\;\;\; e \;\;\;\;\;\;\;\;\!\!\!\!\!\!\longrightarrow$

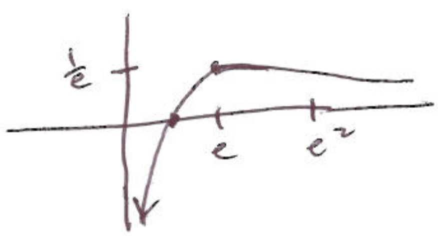
f' $+$ e $-$

f inc $\nearrow$ Dec $\searrow$

$\therefore$ $x = e$ is local Max.

iii $f(1) = \frac{\ln 1}{1} = 0$, $f(\tfrac{1}{2}) = \frac{\ln\frac{1}{2}}{\frac{1}{2}} = 2(\ln 1 - \ln 2) = -2\ln 2$

Graph



$f(e) = \frac{\ln e}{e} = \frac{1}{e}$

$\therefore$ Range: $y \le \frac{1}{e}$

b. $f''(x) = \frac{-\frac{1}{x}\cdot x^2 - (1-\ln x)\cdot 2x}{x^4} = \frac{-x-2x+2x\ln x}{x^4} = \frac{-3x+2x\ln x}{x^4}$

$f''(x) = \frac{\cancel{x}(-3+2\ln x)}{\cancel{x^4}} = \frac{-3+2\ln x}{x^3}$

$f(e^{1.5}) = \frac{\ln e^{1.5}}{e^{1.5}} = \frac{1.5}{e^{1.5}}$

$-3 + 2\ln x = 0$

$2\ln x = 3$

$\ln x = 1.5$

$\boxed{x = e^{1.5} \text{ c.v.}}$

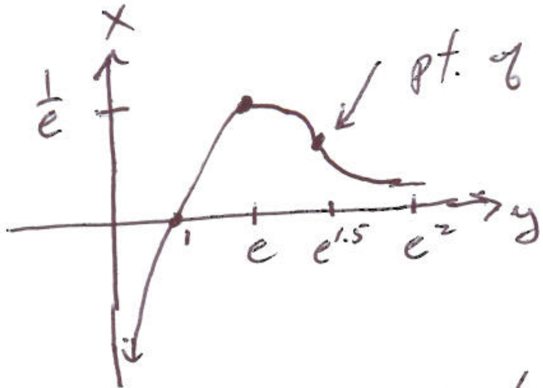
$\longleftarrow \;\; e^{1.5} \;\; \longrightarrow$

f'' $-$ $e^{1.5}$ $+$

f $\cap$ $\cup$

Since there is a sign change pt of inf. $= \left(e^{1.5}, \frac{1.5}{e^{1.5}}\right)$

12 c.



V.A. $x=0$, x-int: $x=1$, local Max: $(e, \frac{1}{e})$

d. Graph $y=\frac{\ln|x|}{x}$, $0<|x|<e^2$

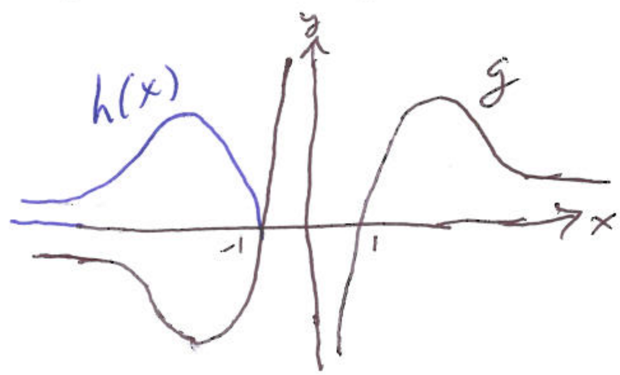
$-e^2<x<e^2$

i Since we know the graph of $y=\frac{\ln x}{x}$ to get $y=\frac{\ln|x|}{x}$

follow these 2 steps:

1) reflection across y-axis

2) reflection across x-axis.



ii Range of g: R

iii Graph $h(x)=\frac{\ln|x|}{|x|}$ follow these steps.

Since $f(x)=\frac{\ln x}{x}$ find graph of $f(|x|)$

$f(|x|)=\frac{\ln|x|}{|x|}$

- leave positive x values alone. (portions above x-axis)
- delete negative x-values. (portions below x-axis)
- reflect across y-axis. (you get an even function)

The graph is superimpose in part i.

So $h(x)>g(x)$ when $-e^2<x<-1$ or $x<-1$

13. Expand $(\cos\theta + i\sin\theta)^3$

a)

$$= \binom{3}{0}\cos^3\theta + \binom{3}{1}\cos^2\theta\,(i\sin\theta) + \binom{3}{2}\cos\theta\,(i\sin\theta)^2 + \binom{3}{3}(i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\boxed{= \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)\,i}$$

b) From De Moivre's theorem

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\therefore \cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)\,i$$

Equating real parts

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= \cos^3\theta + 3\cos^3\theta - 3\cos\theta$$

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$$

13c. $(\cos\theta + i\sin\theta)^5 =$

$= \binom{5}{0}\cos^5\theta + \binom{5}{1}\cos^4\theta\, i\sin\theta + \binom{5}{2}\cos^3\theta(i\sin\theta)^2 + \binom{5}{3}\cos^2\theta(i\sin\theta)^3$

$+ \binom{5}{4}\cos\theta(i\sin\theta)^4 + \binom{5}{5}(i\sin\theta)^5$

$= \cos^5 + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$

$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$

$$\boxed{= \cos^5 - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\, i}$$

$\therefore$ From De Moivre's theorem

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Equating real parts

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$$\boxed{\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$$

13d. Solve $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$
\begin{array}{lr}
 & 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \\
 & 4\cos^3\theta - 3\cos\theta \\
+ & \cos\theta \\
\hline
 & 16\cos^5\theta - 16\cos^3\theta + 3\cos\theta = 0
\end{array}
$$

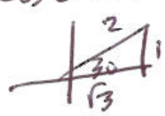
$$\cos\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right) = 0$$

$$\cos\theta\left(4\cos^2\theta - 1\right)\left(4\cos^2\theta - 3\right) = 0$$

$\cos\theta = 0$ or $\cos^2\theta = \frac{1}{4}$ or $\cos^2\theta = \frac{3}{4}$

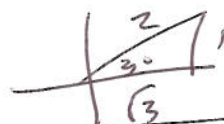
$\boxed{\theta = \frac{\pi}{2}, -\frac{\pi}{2}}$

$\cos\theta = \pm\frac{1}{2}$



$\boxed{\theta = \frac{\pi}{3}, -\frac{\pi}{3}}$

$\cos\theta = \frac{\pm\sqrt{3}}{2}$



$\boxed{\theta = \frac{\pi}{6}, -\frac{\pi}{6}}$

e. $\cos 5\theta = 0$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \ldots$

Consider: $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right) = 0$

Let $w = \cos^2\theta$

$16w^2 - 20w + 5 = 0$

$w = \frac{20 \pm \sqrt{400 - 4\cdot 16\cdot 5}}{32}$

$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4\cdot 16\cdot 5}}{32}$

$\cos\theta = \pm\sqrt{\frac{20 \pm \sqrt{16(25 - 4\cdot 5)}}{32}}$

$\cos\frac{\pi}{10} = Q1 \therefore$ choose + value.

$\cos\frac{\pi}{10} = \sqrt{\frac{20 + 4\sqrt{5}}{32}} =$

$\boxed{\cos\frac{\pi}{10} = \sqrt{\frac{5 + \sqrt{5}}{8}}}$

Consider $\cos\frac{7\pi}{10} = Q2 \therefore \cos$ is negative in $Q2 \therefore$ choose $-$

$\boxed{\cos\frac{7\pi}{10} = -\sqrt{\frac{5 - \sqrt{5}}{8}}}$