M11/5/MATHL/HP2/ENG/TZ1/XX

22117204

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Key

MATHEMATICS
HIGHER LEVEL
PAPER 2

Thursday 5 May 2011 (morning)

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

2211-7204

0114

14 pages

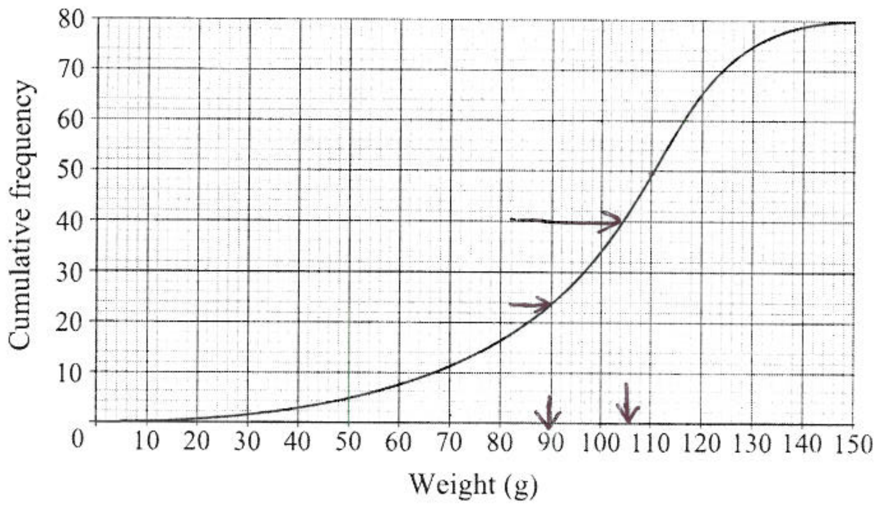
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 4]*

The cumulative frequency graph below represents the weight in grams of 80 apples picked from a particular tree.



(a) Estimate the

(i) median weight of the apples; 105 grams

(ii) 30^{th} percentile of the weight of the apples. $.3(80)=24 \therefore 90$ g. *[2 marks]*

(b) Estimate the number of apples which weigh more than 110 grams. *[2 marks]*

b) $80-49=31$ apples.

2. *[Maximum mark: 6]*

Consider the function $f(x) = x^3 - 3x^2 - 9x + 10$, $x \in \mathbb{R}$.

(a) Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$. *[4 marks]*

(b) Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line. *[2 marks]*

a) $f'(x) = 3x^2 - 6x - 9$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

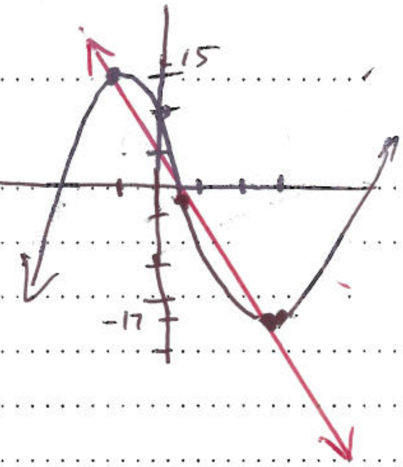
$x = -1, 3$

Min $(3, -17)$, Max $(-1, 15)$

$m = \frac{15+17}{-1-3} = \frac{32}{-4} = -8$

$y - y_1 = m(x - x_1)$

$y - 15 = -8(x+1)$



b) $f''(x) = 6x - 6 = 0$

$x = 1$ c.v.

f'' −   +

f ∩   ∪

Point of inflection! $(1, -1)$

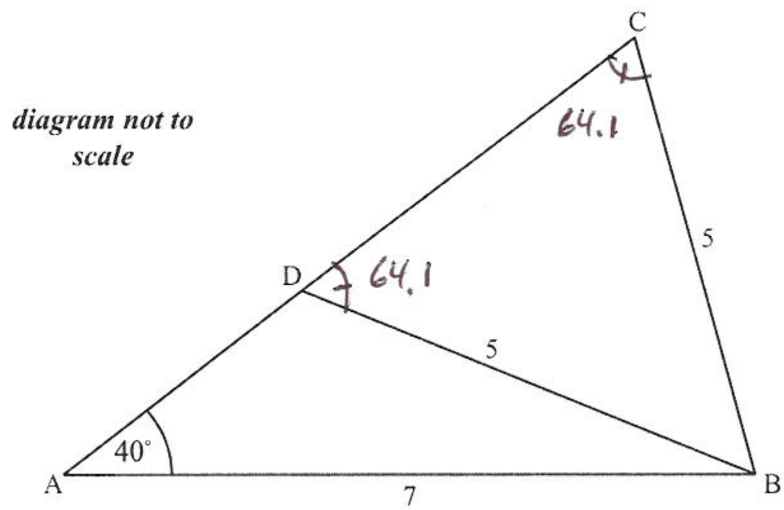
$-1 - 15 = -8(1+1)$

$-16 = -16$ yes.

$\therefore (1, -1)$ lies on the line.

3. *[Maximum mark: 5]*

Given ΔABC, with lengths shown in the diagram below, find the length of the line segment [CD].



$\frac{\sin C}{7} = \frac{\sin 40}{5}$

$C \approx 64.145\ldots = 64.1$ round to 3 S.F.

$\angle CBD = 180 - 2(64.145) = 51.7$

$\frac{CD}{\sin 51.7} = \frac{5}{\sin 64.1}$

$CD = 4.36$ round to 3 S.F.

4. *[Maximum mark: 5]*

The function $f(x) = 4x^3 + 2ax - 7a$, $a \in \mathbb{R}$, leaves a remainder of -10 when divided by $(x - a)$.

(a) Find the value of a. *[3 marks]*

(b) Show that for this value of a there is a unique real solution to the equation $f(x) = 0$. *[2 marks]*

a) $X - a = 0 \rightarrow X = a$

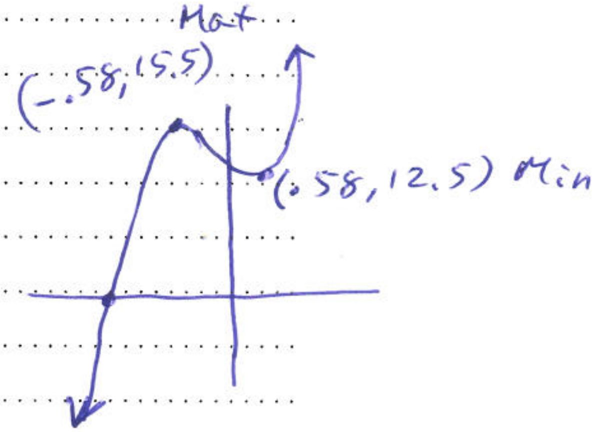
$f(a) = 4a^3 + 2a^2 - 7a = -10$

$= 4a^3 + 2a^2 - 7a + 10 = 0$

From Calculator: $a = -2$

b) Sub $a = -2$ in $f(x)$

$f(x) = 4x^3 - 4x + 14 = 0$



There is one real root @ $X = -1.74$
The other 2 roots are non real.

5. *[Maximum mark: 5]*

(a) Write down the quadratic expression $2x^2 + x - 3$ as the product of two linear factors. *[1 mark]*

(b) Hence, or otherwise, find the coefficient of x in the expansion of $(2x^2 + x - 3)^8$. *[4 marks]*

a) $(2x+3)(x-1)$

b) $[(2x+3)(x-1)]^8 = (2x+3)^8 (x-1)^8$

$\binom{8}{0}(2x)^8(3)^0 \quad \cdot \quad \binom{8}{0}(x)^8(-1)^0$

$\binom{8}{1}(2x)^7(3)^1 \quad \cdot \quad \binom{8}{1}(x)^7(-1)^1$

$\vdots \qquad\qquad \vdots$

$\binom{8}{7}(2x)^1(3)^7 \quad \cdot \quad \binom{8}{7}(x)^1(-1)^7$

$\binom{8}{8}(2x)^0(3)^8 \quad \cdot \quad \binom{8}{8}(x)^0(-1)^8$

To Get only the linear X.

$= \binom{8}{7}(2x)^1(3)^7 \cdot \binom{8}{8}(x)^0(-1)^8 + \binom{8}{8}(2x)^0(3)^8 \cdot \binom{8}{7}(x)^1(-1)^7$

$= 8(2x)(2187) \cdot 1(1)(1) + (1)(1)(6561)(8)(x)(-1)$

$= 34992x + -52488x$

$= -17496x$

$\therefore$ coeffient of x = $\boxed{-17496}$

0614

6. *[Maximum mark: 7]*

The radius of the circle with centre C is 7 cm and the radius of the circle with centre D is 5 cm. If the length of the chord [AB] is 9 cm, find the area of the shaded region enclosed by the two arcs AB.

Note: Use store feature.
1.396444947 enter
sto → C
↳ Alpha prgm.

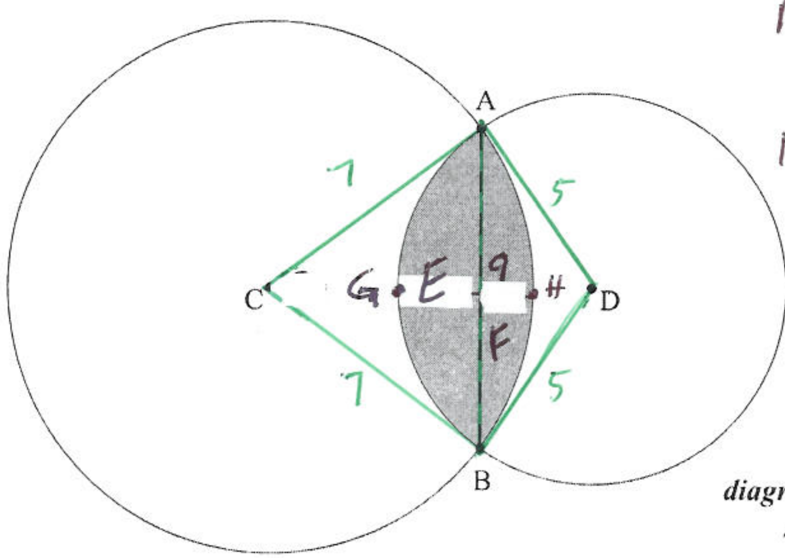


diagram not to scale

$\angle C = \cos^{-1}\left(\frac{7^2+7^2-9^2}{2\cdot 7\cdot 7}\right) = 1.396444947$

$\angle D = \cos^{-1}\left(\frac{5^2+5^2-9^2}{2\cdot 5\cdot 5}\right) = 2.23953903$

Area of $\triangle ADB = \frac{1}{2}\cdot 5\cdot 5 \sin(2.239) = 9.807522623$

Area of sector $AGBD = \frac{1}{2}(2.239)5^2 = 27.99423788$

Shaded area E = Area of sector AGBD − Area of $\triangle ABD$
= 18.18671525

Area of $\triangle ACB = \frac{1}{2}\cdot 7\cdot 7 \sin 1.396 = 24.12856191$

Area of sector $AHBC = \frac{1}{2}(1.396)\cdot 7^2 = 34.2129012$

Shaded area F = sector AHBC − $\triangle$ ACB
= 10.0843393

Shaded area E + Shaded area F = 28.27105455

$= 28.3\ cm^2$ round to 3 S.F.

7. *[Maximum mark: 7]*

A continuous random variable X has a probability density function given by the function $f(x)$, where

$$f(x)=\begin{cases} k(x+2)^2, & -2 \le x < 0 \\ k, & 0 \le x \le \frac{4}{3} \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find the value of k. *[2 marks]*

(b) Hence find

(i) the mean of X; See IB Formula : $E(X)=\int_{-\infty}^{\infty} x\, f(x)\,dx$

(ii) the median of X. *[5 marks]*

a) $k\int_{-2}^{0}(x+2)^2\,dx+\int_{0}^{4/3}k\,dx=1$

$\int x^2+4x+4$

$k\left(\frac{x^3}{3}+2x^2+4x\right)\Big|_{-2}^{0} \quad + kx\Big|_{0}^{4/3}$

$k\left(0-\left(-\frac{8}{3}+8-8\right)\right) \quad + \frac{4}{3}k-0$

$\frac{8}{3}k+\frac{4}{3}k=1$

$\boxed{k=\frac{1}{4}}$

b) i $E(X)=\frac{1}{4}\int_{-2}^{0}x(x+2)^2\,dx+\int_{0}^{4/3}\frac{1}{4}x\,dx$

$=\frac{1}{4}\int x^3+4x^2+4x\,dx+\frac{1}{4}\frac{x^2}{2}\Big|_{0}^{4/3}$

$=\frac{1}{4}\left(\frac{x^4}{4}+\frac{4}{3}x^3+2x^2\right)\Big|_{-2}^{0}$

$=\frac{1}{4}\left(0-\left(4-\frac{32}{3}+8\right)\right)+\frac{1}{8}\left(\frac{16}{9}-0\right)$

$=-\frac{1}{3}+\frac{2}{9}=\boxed{-\frac{1}{9}}$

ii) Median given by some constant a such that $P(X<a)=.5$

$\frac{1}{4}\int_{-2}^{a}(x+2)^2\,dx=.5 \rightarrow \frac{(a+2)^3}{3}-0=2 \rightarrow (a+2)^3=6$

$\frac{(x+2)^3}{3}\Big|_{-2}^{a}=2$

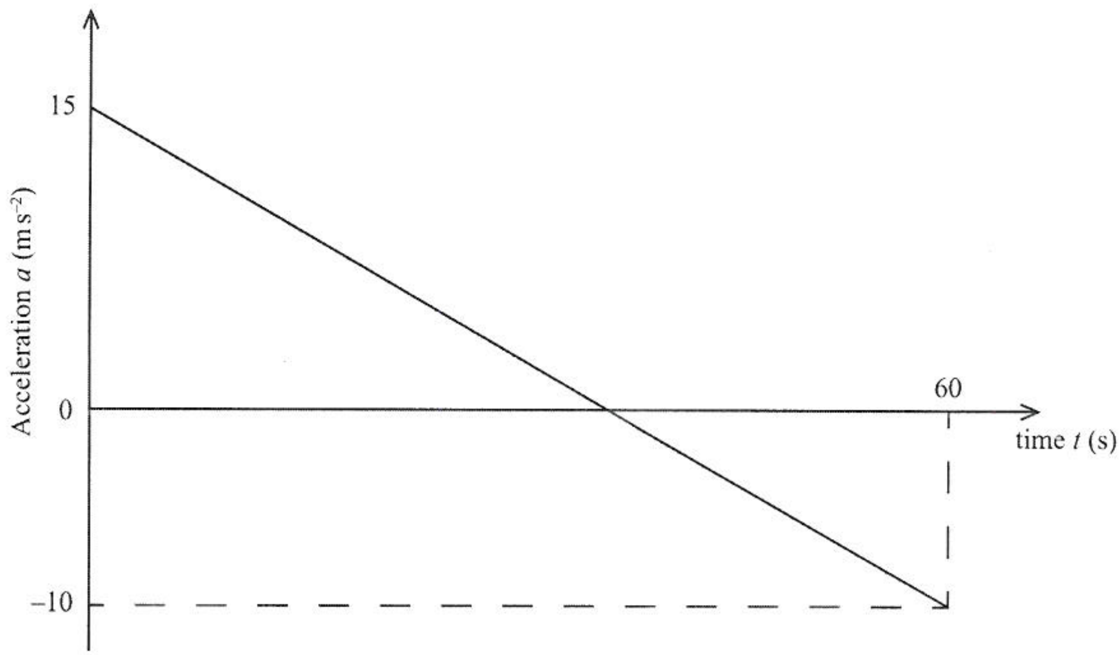
$a+2=\sqrt[3]{6}$

$a=\sqrt[3]{6}-2$ Median

$a=-.183$

8. *[Maximum mark: 8]*

A jet plane travels horizontally along a straight path for one minute, starting at time $t = 0$, where t is measured in seconds. The acceleration, a, measured in m s^{-2}, of the jet plane is given by the straight line graph below.



(a) Find an expression for the acceleration of the jet plane during this time, in terms of t. *[1 mark]*

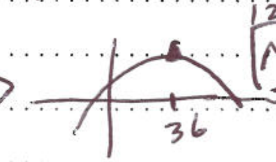
(b) Given that when $t = 0$ the jet plane is travelling at $125\ \text{m s}^{-1}$, find its maximum velocity in m s^{-1} during the minute that follows. *[4 marks]*

(c) Given that the jet plane breaks the sound barrier at $295\ \text{m s}^{-1}$, find out for how long the jet plane is travelling greater than this speed. *[3 marks]*

a) Eq. of line: $a(t) = -\frac{25}{60}t + 15 = -\frac{5}{12}t + 15$

Max @ $395\ \text{m s}^{-1}$

b) $\frac{dv}{dt} = a(t) = -\frac{5}{12}t + 15$ →



$V = \int -\frac{5}{12}t + 15\, dt$

$V = -\frac{5}{24}t^2 + 15t + C$

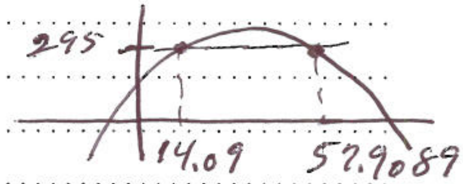
↓

$125 = 0 + 0 + C$

$C = 125$

$V = -\frac{5}{24}t^2 + 15t + 125$

c)



∴ $t = 57.9 - 14.1$

$t = 43.8$

9. *[Maximum mark: 6]*

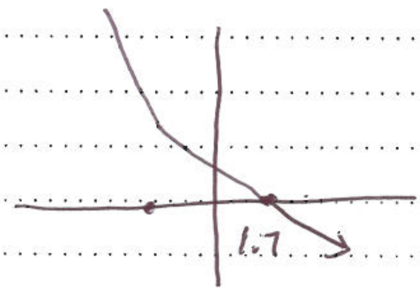
Solve the following system of equations.

$$\log_{x+1} y = 2$$
$$\log_{y+1} x = \frac{1}{4}$$

$(x+1)^2 = y$ and $(y+1)^{1/4} = x$

$(x^2+2x+1+1)^{1/4} = x$

$(x^2+2x+2)^{1/4} - x = 0$



$x = 1.70$ 3 S.F.

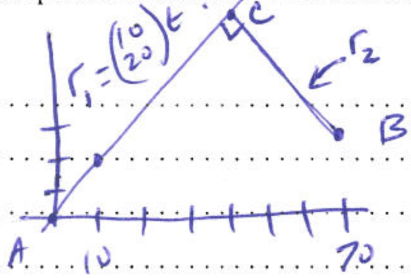
$\therefore (1.7+1)^2 = y$

$y = 7.27$

10. *[Maximum mark: 7]*

Port A is defined to be the origin of a set of coordinate axes and port B is located at the point $(70, 30)$, where distances are measured in kilometres. A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $\boldsymbol{r} = t\begin{pmatrix} 10 \\ 20 \end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?



shortest distance from a point to a line is perpendicular.

$r_2 = \begin{pmatrix} 70 \\ 30 \end{pmatrix} + (t-k)\begin{pmatrix} -60 \\ 30 \end{pmatrix}$

Perpendicular slope is negative direction ∴ −60 in the x-direction

Equate $r_1 = r_2$

$\begin{pmatrix} 10 \\ 20 \end{pmatrix} t = \begin{pmatrix} 70 \\ 30 \end{pmatrix} + (t-k)\begin{pmatrix} -60 \\ 30 \end{pmatrix}$

$10t = 70 + (t-k)(-60) \rightarrow 10t = 70 - 60t + 60k$

$20t = 30 + (t-k)(30) \rightarrow 20t = 30 + 30t - 30k$

$\therefore 70t - 60k = 70 \rightarrow 70t - 60k = 70$

$7(-10t + 30k = 30 \rightarrow) -70t + 210k = 210$

$150k = 280$

$k = 1.8\bar{6}$ hour

$.8\bar{6} \times 60 = 52$ min.

∴ 1 hr 52 min after 10:00

∴ 11:52

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 14]*

The equations of three planes, are given by

$$\begin{aligned} ax + 2y + z &= 3 \\ -x + (a+1)y + 3z &= 1 \\ -2x + y + (a+2)z &= k \end{aligned}$$

where $a \in \mathbb{R}$.

(a) Given that $a = 0$, show that the three planes intersect at a point. *[3 marks]*

(b) Find the value of a such that the three planes do **not** meet at a point. *[5 marks]*

(c) Given a such that the three planes do **not** meet at a point, find the value of k such that the planes meet in one line and find an equation of this line in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

[6 marks]

11a. $\begin{vmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{vmatrix} \Rightarrow det = 0(2-3) - 2(-2+6) + 1(-1+2) = -7$

Since $det \neq 0$ there is a unique solution.

(1) $2y + z = 3$

(2) $-x + y + 3z = 1$

(3) $-2x + y + 2z = K$

Row 3 − 2 times R1 $\Rightarrow -2x - 3y = K - 6 \rightarrow$

Row 2 − 3 times R1 $\Rightarrow (-x - 5y = -8) \cdot 2 \rightarrow$

$$-2x - 3y = K - 6$$
$$2x + 10y = 16$$
$$7y = K + 10$$

$$\boxed{y = \frac{K+10}{7}}$$

$\therefore 2\left(\frac{K+10}{7}\right) + z = 3$

$2K + 20 + 7z = 21$

$7z = 1 - 2K$

$$\boxed{z = \frac{1-2K}{7}}$$

$\therefore 7 \cdot \left[-x + \frac{K+10}{7} + 3\left(\frac{1-2K}{7}\right)\right] = [1]\,7$

$-7x + K + 10 + 3 - 6K = 7$

$7x = -5K + 6$

$$\boxed{x = \frac{6-5K}{7}}$$

The 3 planes intersect @ $\left(\frac{6-5K}{7}, \frac{K+10}{7}, \frac{1-2K}{7}\right)$

11b. $\begin{vmatrix} a & 2 & 1 \\ -1 & a+1 & 3 \\ -2 & 1 & a+2 \end{vmatrix} = a\big((a+1)(a+2)-3\big) - 2\big(-1(a+2)+6\big) + 1\big(-1+2(a+1)\big)$

for planes do not intersect det $= 0$.

$$a(a^2+3a+2-3) - 2(-a+4) + 1(2a+1) = 0$$
$$a^3+3a^2-a+2a-8+2a+1=0$$
$$a^3+3a^2+3a-7=0$$

From Calc: $a=1$

c. when $a=1$

$$\left|\begin{matrix} 1 & 2 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ -2 & 1 & 3 & K \end{matrix}\right| \begin{matrix} R_1+R_2 \\ 2R_1+R_3 \end{matrix} = \left|\begin{matrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 5 & 5 & 6+K \end{matrix}\right| \begin{matrix} \frac{1}{4}R_2 \\ \frac{1}{5}R_3 \end{matrix} = \left|\begin{matrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \frac{6+K}{5} \end{matrix}\right|$$

$$\left|\begin{matrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1-\frac{6+K}{5} \end{matrix}\right| R_2-R_3$$

$\therefore\ 0x+0y+0z = 1-\frac{6+K}{5}$

for infinite # of solutions to exist

$$0 = 1-\frac{6+K}{5}$$
$$\frac{6+K}{5} = 1$$
$$6+K=5$$
$$\boxed{K=-1}$$

$$x+2y+z=3$$
$$y+z=1$$

let x, y, z be any value to find a point.

Ex. let $y=0$ $\therefore z=1$ and $x=2$

So $\begin{pmatrix}2\\0\\1\end{pmatrix}$ is on line.

or let $z=0$ $\therefore y=1$ and $x=1$

So $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is another point on line.

To find direction of line, same as slope, that is, find position vector, $\begin{pmatrix}2\\0\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\therefore$ $\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}}$ Not unique!

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

12. *[Maximum mark: 17]*

A student arrives at a school X minutes after 08:00, where X may be assumed to be normally distributed. On a particular day it is observed that 40 % of the students arrive before 08:30 and 90 % arrive before 08:55.

(a) Find the mean and standard deviation of X. *[5 marks]*

(b) The school has 1200 students and classes start at 09:00. Estimate the number of students who will be late on that day. *[3 marks]*

(c) Maelis had not arrived by 08:30. Find the probability that she arrived late. *[2 marks]*

At 15:00 it is the end of the school day and it is assumed that the departure of the students from school can be modelled by a Poisson distribution. On average 24 students leave the school every minute.

(d) Find the probability that at least 700 students leave school before 15:30. *[3 marks]*

(e) There are 200 days in a school year. Given that Y denotes the number of days in the year that at least 700 students leave before 15:30, find

(i) $\mathrm{E}(Y)$;

(ii) $\mathrm{P}(Y > 150)$. *[4 marks]*

12a. use 2nd vars → invNorm(.4) = -.253... ANS → A Store A
Probability ↲ ↳ z score

invNorm(.9) = 1.28... ⟹ ANS → store B
↳ z-score.

$Z = \frac{X - \mu}{\sigma}$ ⟹ $-.253... = \frac{30 - \mu}{\sigma}$
⟹ $1.28... = \frac{55 - \mu}{\sigma}$

} Solve for σ and equate to find μ.

Note: 8:30 use 30 minutes
8:55 use 55 minutes.

$\sigma = \frac{30 - \mu}{-.253...}$
$\sigma = \frac{55 - \mu}{1.28...}$

∴ $\frac{30 - \mu}{-.253...} = \frac{55 - \mu}{1.28...}$

$38.446547 - 1.28...\mu = -13.934 + .253..\mu$

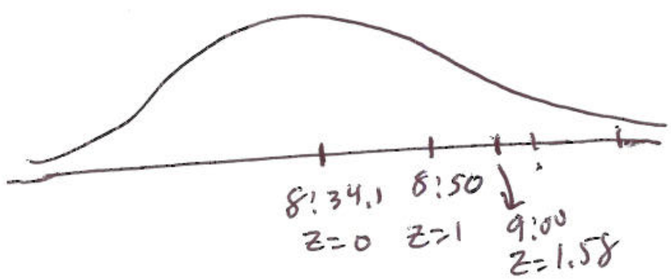
$52.38... = 1.533...\mu$

$\mu = 34.168$

$\mu = 34.1$ 3 S.F.

∴ $\sigma \approx 16.2746$
$\sigma = 16.3$ 3 S.F.

b.



$Z = \frac{60 - 34.1}{16.3}$

$Z = 1.58895$ or $Z = 1.59$ 3 S.F.

Look @ table when $Z = 1.59$ and find P.

$P = .9441$

$.9441 \times 1200 = 1132.92$

$1200 - 1132.92 = 67.08$

or 67 students

12c. $\frac{P(X>60 \mid X>30)}{P(X>30)} = \frac{P(X>60 \cap X>30)}{P(X>30)} = \frac{P(X>60)}{P(X>30)}$

From part b! $P<60 = .9441$

$\therefore P>60 = 1-.9441 = .0559$

From part a: $P<30 = .4$ Given as 40%.

$\therefore P>30 = 1-.4 = .6$

$= \frac{.0559}{.6} = .0931\overline{6}$ round to $\boxed{.093 \text{ 3 S.F.}}$

d. Let X be the random variable of the # of students who leave school in a 30 min. interval.

since $24 \cdot 30 = 720$

$X \sim P_o(m) \Rightarrow X \sim P_o(720)$

$P(X \geq 700) = 1 - P(X < 699) = \boxed{.777}$

ei $Y \sim B(200, .7767)$

$E(Y) = 200 \cdot .7767 = 155$

ii $P(Y > 150) = 1 - P(Y < 150)$

$= \boxed{.797}$

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

13. *[Maximum mark: 13]*

(a) Given that $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, show that $A^2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$. *[3 marks]*

(b) Prove by induction that

$$A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}, \text{ for all } n \in \mathbb{Z}^+.$$ *[7 marks]*

(c) Given that A^{-1} is the inverse of matrix A, show that the result in part (b) is true where $n = -1$. *[3 marks]*

14. *[Maximum mark: 16]*

An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, 2π radians about the y-axis. Units on the coordinate axes are defined to be in centimetres.

(a) When the glass contains water to a height h cm, find the volume V of water in terms of h. *[3 marks]*

(b) If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that,

$$\frac{dV}{dt} = -3\sqrt{2\pi V}, \text{ where } t \text{ is measured in hours.}$$ *[6 marks]*

(c) If the glass is filled completely, how long will it take for all the water to evaporate? *[7 marks]*

13a.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{pmatrix}$$

$\cos^2\theta - \sin^2\theta \rightarrow \cos^2\theta + \cos^2\theta - 1 \rightarrow 2\cos^2\theta - 1 \rightarrow \cos 2\theta$

$\cos\theta\sin\theta + \sin\theta\cos\theta \rightarrow 2\sin\theta\cos\theta \rightarrow \sin 2\theta$

$-\sin\theta\cos\theta - \sin\theta\cos\theta \rightarrow -2\sin\theta\cos\theta \rightarrow -\sin 2\theta$

$-\sin^2\theta + \cos^2\theta \rightarrow \cos^2\theta - 1 + \cos^2\theta \rightarrow 2\cos^2\theta - 1 \rightarrow \cos 2\theta$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

b. Let $P(n)$ be the proposition that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$ for all $n \in \mathbb{Z}^+$

$P(1)$ is true $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Assume $P(K)$ is true.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^K = \begin{pmatrix} \cos K\theta & \sin K\theta \\ -\sin K\theta & \cos K\theta \end{pmatrix}$$

Consider $P(K+1)$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{K+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^K \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^1$$

$$= \begin{pmatrix} \cos K\theta & \sin K\theta \\ -\sin K\theta & \cos K\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos K\theta\cos\theta - \sin K\theta\sin\theta & \cos K\theta\sin\theta + \sin K\theta\cos\theta \\ -\sin K\theta\cos\theta - \cos K\theta\sin\theta & -\sin K\theta\sin\theta + \cos K\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(K+1)\theta & \sin(K+1)\theta \\ -\sin(K+1)\theta & \cos(K+1)\theta \end{pmatrix}$$

If $P(K)$ is true then $P(K+1)$ is true since $P(1)$ is true then $P(n)$ is true for all $n \in \mathbb{Z}^+$

QIV cos is +   QIV sin is −

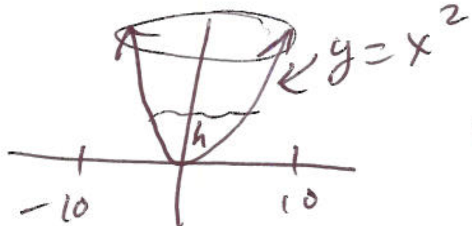
13c. $A^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$

$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A \cdot A^{-1}$ = Identity. ∴ A^{-1} is inverse of A.

14a.



$$V = \pi \int_0^h (\sqrt{y})^2 \, dy$$

$$= \pi \int_0^h y \, dy = \frac{\pi}{2} y^2 \Big|_0^h = \boxed{\frac{\pi h^2}{2} = V}$$

b. $\frac{dV}{dt} = -3$ times surface area.

↳ $S.A = \pi x^2$
$= \pi h$

$\frac{dV}{dt} = -3\pi h$, Given $V = \frac{\pi h^2}{2}$

$\therefore h = \sqrt{\frac{2V}{\pi}}$

$$\frac{dV}{dt} = -3\pi\left(\sqrt{\frac{2V}{\pi}}\right)$$

$$= -3\pi \frac{\sqrt{2V}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$= -3\pi \frac{\sqrt{2\pi V}}{\pi} \Rightarrow \boxed{\frac{dV}{dt} = -3\sqrt{2\pi V}}$$

c. $V = \frac{\pi h^2}{2}$, when $x = 10$, $y = 100$, so $h = 100$

$V = \frac{\pi (100)^2}{2} = 5000\pi$ This is how much water glass has when it is completely filled.

Separate variables:

$$\frac{dV}{dt} = -3\sqrt{2\pi} \cdot \sqrt{V}$$

$$\int \frac{dV}{\sqrt{V}} = \int -3\sqrt{2\pi} \, dt$$

$$\int V^{-1/2} dV = -3\sqrt{2\pi}\, t + C$$

→ $2\sqrt{V} = -3\sqrt{2\pi}\, t + C$

when $V = 5000\pi$, $t = 0$

$\therefore 2\sqrt{5000\pi} = C$

$$\boxed{2\sqrt{V} = -3\sqrt{2\pi}\, t + 2\sqrt{5000\pi}}$$

when $V = 0$, empty of water.

$0 = -3\sqrt{2\pi}\, t + 2\sqrt{5000\pi}$

$$\boxed{t = 33\tfrac{1}{3} \text{ hours}}$$