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MATHEMATICS
STANDARD LEVEL
PAPER 1

Walbel Key

Candidate session number

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Wednesday 4 May 2011 (afternoon)

Examination code

2	2	1	1	–	7	3	0	3

1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** questions in the boxes provided.*

1. *[Maximum mark: 5]*

Let $f(x) = 7 - 2x$ and $g(x) = x + 3$.

(a) Find $(g \circ f)(x)$. *[2 marks]*

(b) Write down $g^{-1}(x)$. *[1 mark]*

(c) Find $(f \circ g^{-1})(5)$. *[2 marks]*

a) $(g \circ f)(x) = g(f(x)) = g(7-2x)$

$= 7-2x+3 = \boxed{10-2x}$

b) $y = x+3$

$x = y+3$ interchange x for y and solve for y.

$\boxed{y^{-1} = g^{-1}(x) = x-3}$

c) $(f \circ g^{-1})(5) = f(g^{-1}(5)) = f(5-3) = f(2)$

$= 7-2(2) = 7-4 = \boxed{3}$

2. *[Maximum mark: 6]*

A line L passes through $A(1, -1, 2)$ and is parallel to the line $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$.

(a) Write down a vector equation for L in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$. *[2 marks]*

The line L passes through point P when $t = 2$.

(b) Find

(i) $\overrightarrow{OP}$;

(ii) $|\overrightarrow{OP}|$. IB Packet for formula *[4 marks]*

a) $r = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

b i) $\overrightarrow{OP} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$

b ii) $|\overrightarrow{OP}| = \sqrt{3^2 + 5^2 + (-2)^2}$

$= \sqrt{9 + 25 + 4}$

$\boxed{= \sqrt{38}}$

3. *[Maximum mark: 6]*

Let $A = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$.

(a) Find A^{-1}. See IB formula. *[2 marks]*

(b) Solve the matrix equation $AX = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}$. *[4 marks]*

a) $A^{-1} = \frac{1}{2\cdot 3 - -4\cdot -1}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 2 \\ 1/2 & 1 \end{pmatrix}$

b) $X = A^{-1} \cdot \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\cdot 4 + 2\cdot 2 & \frac{3}{2}\cdot 6 + 2\cdot -2 \\ \frac{1}{2}\cdot 4 + 1\cdot 2 & \frac{1}{2}\cdot 6 + 1\cdot -2 \end{pmatrix}$

$= \begin{pmatrix} 10 & 5 \\ 4 & 1 \end{pmatrix}$

4. *[Maximum mark: 7]*

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X=x)=\frac{x^2}{14},\ x\in\{1,2,k\},\text{ where } k>0.$$

(a) Write down $\mathrm{P}(X=2)$. *[1 mark]*

(b) Show that $k=3$. *[4 marks]*

(c) Find $\mathrm{E}(X)$. $E(X)=\sum x\,P(X=x)$ See IB formula *[2 marks]*

a) $P(X=2)=\frac{2^2}{14}=\frac{4}{14}=\frac{2}{7}$

b) $\frac{1^2}{14}+\frac{2^2}{14}+\frac{K^2}{14}=1$

$\frac{1}{14}+\frac{4}{14}+\frac{K^2}{14}=\frac{14}{14}$

$K^2=9$

$K=\pm 3$ since $K>0$ $\boxed{K=3}$

c)

x	1	2	3
$P(X=x)$	$\frac{1}{14}$	$\frac{4}{14}$	$\frac{9}{14}$

$E(x)=1\cdot\frac{1}{14}+2\cdot\frac{4}{14}+3\cdot\frac{9}{14}=\frac{1}{14}+\frac{8}{14}+\frac{27}{14}$

$=\frac{36}{14}=\boxed{\frac{18}{7}}$

5. *[Maximum mark: 7]*

Let $g(x) = \frac{\ln x}{x^2}$, for $x > 0$.

See IB Packet

(a) Use the quotient rule to show that $g'(x) = \frac{1 - 2\ln x}{x^3}$. *[4 marks]*

(b) The graph of g has a maximum point at A. Find the x-coordinate of A. *[3 marks]*

a) $g'(x) = \frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{(x^2)^2} = \frac{x - 2x\ln x}{x^4}$

$= \frac{x(1 - 2\ln x)}{x^4} = \boxed{\frac{1 - 2\ln x}{x^3}}$

b) Set $g'(x) = 0$ and solve for x.

$\frac{1 - 2\ln x}{x^3} = 0$

$1 - 2\ln x = 0$

$2\ln x = 1$

$e^{[\ln x]} = e^{[\frac{1}{2}]}$

$\boxed{x = e^{1/2} = \sqrt{e}}$

6. *[Maximum mark: 7]*

Solve the equation $2\cos x = \sin 2x$, for $0 \le x \le 3\pi$.

$\sin 2x = 2\sin x \cos x$
IB Packet.

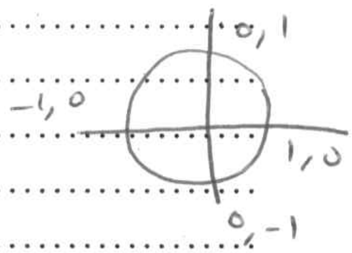
$2\cos x - \sin 2x = 0$

$2\cos x - 2\sin x \cos x = 0$

$2\cos x(1 - \sin x) = 0$

$2\cos x = 0$ or $1 - \sin x = 0$

$\cos x = 0$ $\sin x = 1$



$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ $x = \frac{\pi}{2}, \frac{5\pi}{2}$

So $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

7. *[Maximum mark: 7]*

Consider $f(x) = 2kx^2 - 4kx + 1$, for $k \neq 0$. The equation $f(x) = 0$ has two equal roots.

(a) Find the value of k. *[5 marks]*

(b) The line $y = p$ intersects the graph of f. Find all possible values of p. *[2 marks]*

a) Set Discriminant $b^2 - 4ac = 0$

$(-4k)^2 - 4(2k)(1) = 0$

$16k^2 - 8k = 0$

$8k(2k-1) = 0$

$8k = 0$ or $2k - 1 = 0$

$k \neq 0$ or $k = \frac{1}{2}$

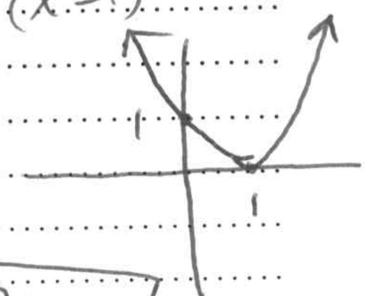
$\boxed{k = \frac{1}{2}}$

b) $f(x) = 2 \cdot \frac{1}{2}x^2 - 4\frac{1}{2}x + 1$

$= x^2 - 2x + 1$

$(x-1)(x-1)$

$(x-1)^2$



$\boxed{p \geq 0}$

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** questions on the answer sheets provided. Please start each question on a new page.*

8. *[Maximum mark: 13]*

In a group of 16 students, 12 take art and 8 take music. One student takes neither art nor music. The Venn diagram below shows the events art and music. The values p, q, r and s represent numbers of students.

$n(A \cup M) = n(A) + n(M) - n(A \cap M)$

$15 = 12 + 8 - n(A \cap M)$

$A \cap M = 20 - 15$

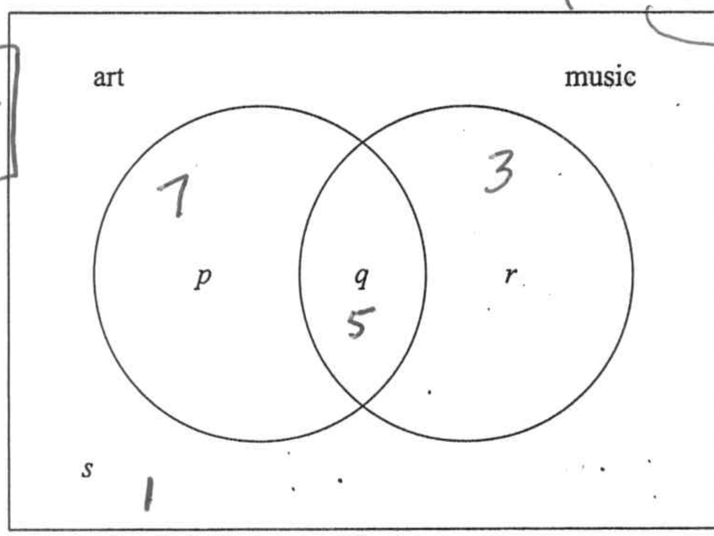
$\boxed{n(A \cap M) = 5}$

$\boxed{q = 5}$

8c)

$\frac{3}{16} \cdot \frac{7}{16} = \boxed{\frac{21}{256}}$

16×16: 96, 16, 256



(a) (i) Write down the value of s. $16 - 1 = 15$

(ii) Find the value of q. 5

(iii) Write down the value of p and of r. $p = 12 - 5 = 7$, $r = 8 - 5 = 3$ *[5 marks]*

(b) (i) A student is selected at random. Given that the student takes music, write down the probability the student takes art.

(ii) **Hence**, show that taking music and taking art are **not** independent events. *[4 marks]*

(c) Two students are selected at random, one after the other. Find the probability that the first student takes **only** music and the second student takes **only** art. *[4 marks]*

bi) $P(A \mid M) = \frac{P(A \cap M)}{P(M)} = \frac{\frac{5}{16}}{\frac{12}{16}} = \frac{5}{16} \cdot \frac{16}{12} = \boxed{\frac{5}{12}}$

bii) $P(A \cap M) = P(A) \cdot P(M)$ If they are independent.

$\frac{5}{12} = \frac{12}{16} \cdot \frac{8}{16} \rightarrow \frac{5}{12} = \frac{3}{8}$

$\frac{10}{24} \neq \frac{9}{24}$ ∴ not independents.

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

9. *[Maximum mark: 16]*

The following diagram shows the obtuse-angled triangle ABC such that $\vec{AB} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$.

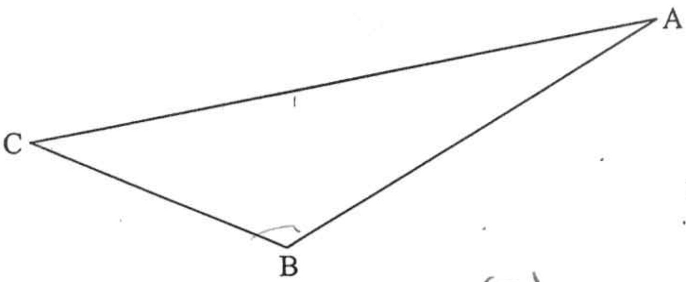


diagram not to scale

(a) (i) Write down $\vec{BA}$. $\vec{BA} = -\vec{AB} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

(ii) Find $\vec{BC}$. $\vec{BC} = \vec{AC} - \vec{AB} = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ *[3 marks]*

(b) (i) Find $\cos A\hat{B}C$.

(ii) Hence, find $\sin A\hat{B}C$. *[7 marks]*

The point D is such that $\vec{CD} = \begin{pmatrix} -4 \\ 5 \\ p \end{pmatrix}$, where $p > 0$.

c i) $|CD| = \sqrt{4^2 + 5^2 + p^2} = \sqrt{50}$

$16 + 25$

$41 + p^2 = 50$

$p^2 = 9$

$p = \pm 3$

Since $p > 0$ so $\boxed{p = 3}$

(c) (i) Given that $|\vec{CD}| = \sqrt{50}$, show that $p = 3$.

(ii) Hence, show that $\vec{CD}$ is perpendicular to $\vec{BC}$. *[6 marks]*

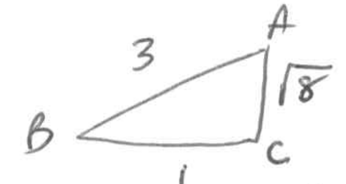
b i) $\cos A\hat{B}C = \dfrac{\vec{AB} \cdot \vec{BC}}{|AB|\,|BC|} = \dfrac{\begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{3^2+0^2+4^2}\,\sqrt{1^2+2^2+2^2}} = \dfrac{-3+0+8}{15} = \dfrac{5}{15} = \boxed{\dfrac{1}{3}}$

$\sqrt{9+16}$ $\sqrt{1+4+4}$

$\sqrt{25}$ $\sqrt{9}$

$5 \cdot 3$

b ii)



so $\sin A\hat{B}C = \boxed{\dfrac{\sqrt{8}}{3}}$

c ii $CD \perp BC$ if $CD \cdot BC = 0$

$\begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -4 + 10 - 6 = \boxed{0}$ yes.

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

10. *[Maximum mark: 16]*

The velocity v m s^{-1} of a particle at time t seconds, is given by $v = 2t + \cos 2t$, for $0 \le t \le 2$.

(a) Write down the velocity of the particle when $t = 0$. $V = 2\cdot 0 + \cos 2\cdot 0$ *[1 mark]*

$V = \cos 0 \Rightarrow \boxed{V = 1}$

When $t = k$, the acceleration is zero.

(b) (i) Show that $k = \frac{\pi}{4}$.

(ii) Find the exact velocity when $t = \frac{\pi}{4}$. *[8 marks]*

(c) When $t < \frac{\pi}{4}$, $\frac{dv}{dt} > 0$ and when $t > \frac{\pi}{4}$, $\frac{dv}{dt} > 0$.

Sketch a graph of v against t. *[4 marks]*

(d) Let d be the distance travelled by the particle for $0 \le t \le 1$.

(i) Write down an expression for d. $d = \int_0^1 |v(t)|\,dt = \int_0^1 |2t + \cos 2t|\,dt$

(ii) Represent d on your sketch. *[3 marks]*

bi) $a = v' = 2 - 2\sin 2t$

$2 - 2\sin 2k = 0$

$2\sin 2k = 2$

$\sin 2k = 1$

$2k = \frac{\pi}{2}$

$\boxed{k = \frac{\pi}{4}}$

c) Range of $\cos 2t = [-1, 1]$

$y_1 = 2t - 1$, $y_2 = 2t + 1$ } $y_1 \parallel y_2$

Period $= \frac{2\pi}{|b|}$

$P = \frac{2\pi}{2}$

$P = \pi$