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International Baccalaureate
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Bachillerato Internacional

Wahbeh Key

Candidate session number

Examination code

2	2	1	4	–	7	2	0	4

MATHEMATICS
HIGHER LEVEL
PAPER 2

Wednesday 14 May 2014 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

15 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** questions in the boxes provided. Working may be continued below the lines if necessary.*

1. *[Maximum mark: 4]*

One root of the equation $x^2+ax+b=0$ is $2+3\text{i}$ where a, $b \in \mathbb{R}$. Find the value of a and the value of b.

Method 1: The product of 2 terms is b.

$\therefore \ (2+3i)(2-3i) = 4-9i^2 = 4+9$

$\boxed{b=13}$

The sum of 2 terms is $-a$ for complex

$(2+3i)+(2-3i) = \boxed{4=-a} \Rightarrow a=-4$

Method 2: Since $X=2+3i$ Sub into eq.

$(2+3i)^2 + a(2+3i)+b=0$

$4+12i+9i^2+2a+3ai+b=0$

$-5+12i+2a+3ai+b=0$

$-5+2a+b+(12+3a)i=0$

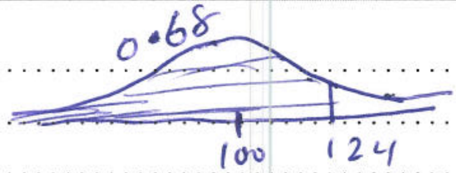
Equate Real w/ real & imaginary w/ imaginary

$\therefore \ 12+3a=0 \Rightarrow 3a=-12 \Rightarrow \boxed{a=-4}$

$-5+2a+b=0 \Rightarrow -5+2(-4)+b=0 \Rightarrow \boxed{b=13}$

2. *[Maximum mark: 5]*

A student sits a national test and is told that the marks follow a normal distribution with mean 100. The student receives a mark of 124 and is told that he is at the 68th percentile. Calculate the variance of the distribution.



$X : N(100, \sigma^2)$, $P(X < 124) = 0.68$

invnormal $(0.68) = 0.4676988012 \xrightarrow{\text{store}} Z$

$Z = \dfrac{x - \mu}{\sigma}$

$Z = \dfrac{124 - 100}{\sigma}$

$\sigma = \dfrac{24}{Z}$

$\sigma = 51.3150$

$Var(x) = \sigma^2 = (51.3150)^2 = \boxed{2630}$

3. *[Maximum mark: 4]*

Find the number of ways in which seven different toys can be given to three children, if the youngest is to receive three toys and the others receive two toys each.

$\binom{7}{3}\binom{4}{2} = \boxed{210}$

4. *[Maximum mark: 6]*

Goal: eliminate one variable and have a system of 2 equations w/ same variables, namely z.

A system of equations is given below.

$$
\begin{aligned}
(1) \quad & x+2y-z=2 \\
(2) \quad & 2x+y+z=1 \\
(3) \quad & -x+4y+az=4
\end{aligned}
$$

(a) Find the value of a so that the system does not have a unique solution. *[4]*

(b) Show that the system has a solution for any value of a. *[2]*

a) $(1)+(3)=(4) \Rightarrow 6y+az-z=6$

$2(1)-(2)=(5) \Rightarrow 3y-3z=3$

$2(5) \Rightarrow 6y-6z=6$

$\ominus \; 6y+az-z=6$

$az+5z=0$

$(a+5)z=0$

If $a=-5$, then $0=0$ and system does not have a unique solution.

b) If $a=-5$, then $0=0$ means that system has infinitely many solutions.

If $a\neq-5$, then there is a unique solution.

hence, there is always a solution.

5. *[Maximum mark: 8]*

The shaded region S is enclosed between the curve $y = x + 2\cos x$, for $0 \le x \le 2\pi$, and the line $y = x$, as shown in the diagram below.

Radian Mode

a). $x + 2\cos x = x$

$2\cos x = 0$

$\cos x = 0$

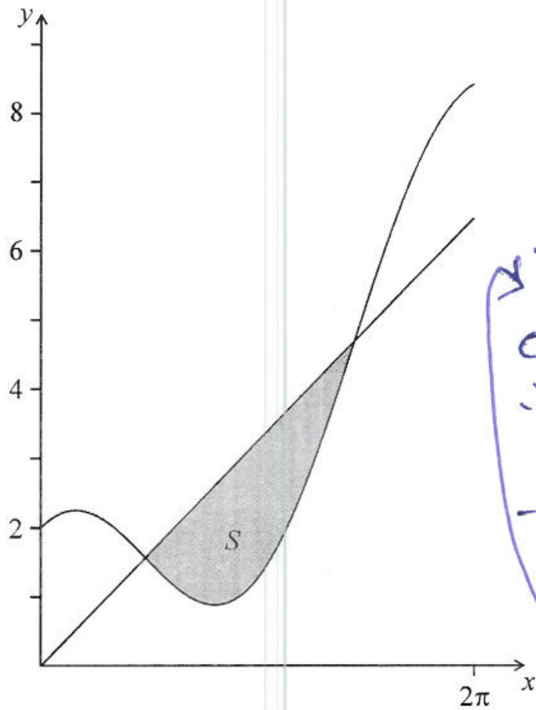
0,1

0,−1

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

Since $y = x$

$\boxed{\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)}$



$-4\int \cos^2 x \, dx$

$\cos 2x = 2\cos^2 x - 1$

$\therefore \cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$

$-4\int \frac{1}{2}\cos 2x + \frac{1}{2} \, dx$

$-4\left(\frac{\sin 2x}{4} + \frac{1}{2}x\right)$

$-\sin 2x - 2x \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

(a) Find the coordinates of the points where the line meets the curve. *[3]*

The region S is rotated by 2π about the x-axis to generate a solid.

(b) (i) Write down an integral that represents the volume V of the solid.

(ii) Find the volume V. *[5]*

Top − Bottom

bi) $V = \pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[(x)^2 - (x + 2\cos x)^2\right] dx$

ii) From GDC $\boxed{V = 18.8\pi \text{ or } 59.2 \text{ 3 S.F.}}$

$V = \pi \int x^2 - (x^2 + 4x\cos x + 4\cos^2 x) \, dx$

$= \pi \int (-4x\cos x - 4\cos^2 x) \, dx$

$-4x \quad \cos x$

$-4 \quad \sin x$

$0 \quad -\cos x$

$-4x\sin x - 4\cos x \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$\boxed{\text{Next Page}}$

$(0,1)$

$(0,-1)$ $(0,-1)$

(Question 5 continued)

$V = \pi\left[-4x\sin x - 4\cos x\right]\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \sin 2x - 2x\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$= \pi\left[\left(-4\frac{3\pi}{2}\sin\frac{3\pi}{2} - 4\cos\frac{3\pi}{2}\right) - \left(-4\frac{\pi}{2}\sin\frac{\pi}{2} - 4\cos\frac{\pi}{2}\right)\right]$

$= \pi\left[(6\pi - 0) - (-2\pi - 0)\right]$

$= \pi\left[6\pi + 2\pi\right]$

$= \pi\left[8\pi\right] \quad + \pi\left[\left(-\sin 2\frac{3\pi}{2} - 2\left(\frac{3\pi}{2}\right)\right) - \left(-\sin 2\frac{\pi}{2} - 2\left(\frac{\pi}{2}\right)\right)\right]$

$= \quad \pi\left[(1 - 3\pi) - (1 - \pi)\right]$

$\pi(-2\pi)$

$= \pi\left[8\pi - 2\pi\right]$

$= \pi(6\pi)$

$= 6\pi^2$

or

59.2

6. *[Maximum mark: 10]*

Let $f(x) = x(x+2)^6$.

(a) Solve the inequality $f(x) > x$. *[5]*

(b) Find $\int f(x)\,\mathrm{d}x$. *[5]*

a) $x(x+2)^6 > x$

$x(x+2)^6 = x$

$x(x+2)^6 - x = 0$

$x[(x+2)^6 - 1] = 0$

$\boxed{x=0}$ or $(x+2)^6 - 1 = 0$

$(x+2)^6 = 1$

$x+2 = \pm 1$

$\boxed{\begin{array}{l} x = -2+1 = -1 \\ x = -2-1 = -3 \end{array}}$

for $x(x+2)^6 > x$

False, True, F, T

$-3 \quad -1 \quad 0$

$\boxed{\begin{array}{c} \therefore\ -3 < x < -1 \\ \text{or} \\ x > 0 \end{array}}$

b) $\int x(x+2)^6\,dx$

$u = x+2 \Rightarrow x = u-2$

$dx = du$

$\int (u-2)u^6\,du$

$\int u^7 - 2u^6\,du$

$\frac{u^8}{8} - \frac{2u^7}{7} + C$

$\boxed{\frac{(x+2)^8}{8} - \frac{2(x+2)^7}{7} + C}$

7. *[Maximum mark: 8]*

Prove, by mathematical induction, that $7^{8n+3}+2$, $n \in \mathbb{N}$, is divisible by 5.

To be divisible by 5 it must end w/ 0 or 5.

If $n=0$

$7^3+2=345$ which is $\div$ by 5, hence true for $n=0$.

Assume true for $n=K$.

$7^{8K+3}+2=5p$, where $p \in$ •

If $n=K+1$, then

$7^{8(K+1)+3}+2$

$7^{8K+8+3}+2$

$7^8 \cdot 7^{8K+3}+2$ Now sub for $5p$

$7^8(5p-2)+2$

$7^8 \cdot 5p-2\cdot 7^8+2$

$7^8 \cdot 5p-11529600$

$5(7^8 p-2305920)$, hence if true for $n=K$, then also true for $n=K+1$.

Since $n=0$ true, then true for all $n \in$ •

8. *[Maximum mark: 8]*

(a) Find the term in x^5 in the expansion of $(3x+A)(2x+B)^6$. *[4]*

Mina and Norbert each have a fair cubical die with faces labelled 1, 2, 3, 4, 5 and 6; they throw it to decide if they are going to eat a cookie.

Mina throws her die just once and she eats a cookie if she throws a four, a five or a six. Norbert throws his die six times and each time eats a cookie if he throws a five or a six.

(b) Calculate the probability that five cookies are eaten. *[4]*

a) $\binom{6}{0}(2x)^6$

$\binom{6}{1}(2x)^5(B) = (192x^5B)A$

$\binom{6}{2}(2x)^4B^2 = (240x^4B^2)(3x) = 720x^5B^2$

You can continue to expand $(2x+B)^6$ Then multiply by $(3X+A)$.

$\therefore 192x^5BA + 720x^5B^2$

$\boxed{= x^5(192BA + 720B^2)}$

b) $P(5 \text{ eaten}) = P(M \text{ eats } 1) \times P(N \text{ eats } 4) + P(M \text{ eats } 0)\, P(N \text{ eats } 5)$

$= \frac{1}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \frac{1}{2}\binom{6}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1$

$\boxed{= \frac{4}{81} = 0.0494}$

Method 2: $x = \frac{1}{6}$, Mina $\boxed{A = \frac{3}{6} = \frac{1}{2}}$, Norbert $\boxed{B = \frac{4}{6} = \frac{2}{3}}$

$= \left(\frac{1}{6}\right)^5\left[192\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + 720\left(\frac{2}{3}\right)^2\right]$

$\boxed{= 0.0494}$

Mean ⇒ 1 day = 5.84 birds
7 days = 40.88

9. *[Maximum mark: 7]*

The number of birds seen on a power line on any day can be modelled by a Poisson distribution with mean 5.84.

$X \sim P_o(5.84)$

(a) Find the probability that during a certain seven-day week, more than 40 birds have been seen on the power line. *[2]*

(b) On Monday there were more than 10 birds seen on the power line. Show that the probability of there being more than 40 birds seen on the power line from that Monday to the following Sunday, inclusive, can be expressed as:

$$\frac{\mathrm{P}(X > 40) + \sum_{r=11}^{40} \mathrm{P}(X = r)\mathrm{P}(Y > 40 - r)}{\mathrm{P}(X > 10)} \text{ where } X \sim \mathrm{Po}(5.84) \text{ and } Y \sim \mathrm{Po}(35.04).$$ *[5]*

a) $X \sim P_o(5.84) \Rightarrow X \sim P_o(40.88)$

$P(X > 40) = 1 - P(X \le 40)$

$1 - \text{poisson}(40.88, 40) = \boxed{0.513}$

b) $\dfrac{\text{Probability more than 10 on Monday AND more than 40 over the week}}{\text{Prob. more than 10 on Monday}}$

Possibilities for numerator are:
more than 40 birds on Monday
11 on M and more than 29 over the course of next 6 days
12 on M 〃 〃 〃 28 〃 ... until
40 on M and more than 0 over the next 6 days

hence, if X is # of birds on power line on M and Y, # on T – Sunday, then numerator is

$P(X > 40) + P(X=11) \cdot P(Y > 29) + P(X=12) \cdot P(Y > 28) + \ldots$
$+ P(X=40) \cdot P(Y > 0)$
$= P(X > 40) + \sum_{r=11}^{40} P(X=r) \cdot P(Y > 40-r)$

hence the solution above.

Turn over

*Do **NOT** write solutions on this page.*

SECTION B

*Answer **all** questions in the answer booklet provided. Please start each question on a new page.*

10. *[Maximum mark: 21]*

Let $f(x) = \dfrac{e^{2x}+1}{e^{x}-2}$.

(a) Find the equations of the horizontal and vertical asymptotes of the curve $y = f(x)$. *[4]*

(b) (i) Find $f'(x)$.

(ii) Show that the curve has exactly one point where its tangent is horizontal.

(iii) Find the coordinates of this point. *[8]*

(c) Find the equation of L_1, the normal to the curve at the point where it crosses the y-axis. *[4]*

The line L_2 is parallel to L_1 and tangent to the curve $y = f(x)$.

(d) Find the equation of the line L_2. *[5]*

a) V.A. Set D=0 $\Rightarrow$ $e^x - 2 = 0 \Rightarrow e^x = 2 \Rightarrow$
$\ln e^x = \ln 2 \Rightarrow \boxed{x = \ln 2}$

H.A. $\Rightarrow \lim\limits_{x\to\infty} \dfrac{e^{2x}+1}{e^x-2} = \dfrac{\infty}{\infty}$ Indeterminate Form use L. Rule.

$\lim\limits_{x\to\infty} \dfrac{2e^{2x}}{e^x} = \dfrac{2e^x \cdot \cancel{e^x}}{\cancel{e^x}} = \lim\limits_{x\to\infty} 2e^x = \infty$, No H.A.

$\lim\limits_{x\to-\infty} \dfrac{e^{2x}+1}{e^x-2} = \dfrac{e^{-\infty}+1}{e^{-\infty}-2} = \dfrac{\frac{1}{e^{\infty}}+1}{\frac{1}{e^{\infty}}-2} = \boxed{-\dfrac{1}{2} = y \text{ H.A.}}$

2014 P2 HL

10 bi $f(x) = \dfrac{e^{2x}+1}{e^x-2}$

$$f'(x) = \frac{2e^{2x}(e^x-2) - e^x(e^{2x}+1)}{(e^x-2)^2}$$

$$= \frac{2e^{3x} - 4e^{2x} - e^{3x} - e^x}{(e^x-2)^2}$$

$$\boxed{f'(x) = \frac{e^{3x} - 4e^{2x} - e^x}{(e^x-2)^2}}$$

bii Show curve has exactly one pt. where its tangent is horizontal.

Set $f'(x) = 0$

$$e^{3x} - 4e^{2x} - e^x = 0$$

$$e^x(e^{2x} - 4e^x - 1) = 0$$

$\boxed{e^x = 0}$ or $e^{2x} - 4e^x - 1 = 0$

let $u = e^x$

$\therefore$ $u^2 - 4u - 1 = 0$

$$u = \frac{4 \pm \sqrt{16 - 4(-1)}}{2}$$

$$u = \frac{4 \pm 2\sqrt{5}}{2}$$

$u = 2 \pm \sqrt{5}$

$\boxed{\therefore e^x = 2 \pm \sqrt{5}}$

For exactly one solution

$\boxed{e^x > 0}$

biii)

$e^x = 2 \pm \sqrt{5}$

$x_1 = \ln(2+\sqrt{5}) = 1.44$

$x_2 = \ln(2-\sqrt{5}) = -1.44$

Min @ $x = 1.44$

$f(1.44) = 8.47$

$\boxed{\therefore (1.44, 8.47)}$

10c Find eq of L_1, normal to curve @ pt. where it crosses y-axis.

$$f(x) = \frac{e^{2x}+1}{e^x-2}$$ Set $x=0$ and solve

$$y = \frac{e^0+1}{e^0-2} \Rightarrow y = \frac{1+1}{1-2} \Rightarrow y = -2$$

$\boxed{\therefore (0,-2) \text{ y-int.}}$

d). $L_2 // L_1$ and tangent to curve $y = f(x)$
Find eq. of L_2.

$$f'(0) = \frac{e^0 - 4e^0 - e^0}{(e^0-2)^2}$$

$$= \frac{1-4-1}{1}$$

$\boxed{f'(0) = -4}$

$\therefore$ Slope of normal $= \frac{1}{4}$

That is $f'(x) = \frac{1}{4}$

$$\frac{1}{4} = \frac{e^{3x}-4e^{2x}-e^x}{(e^x-2)^2}$$

$$y_1 = \frac{e^{3x}-4e^{2x}-e^x}{(e^x-2)^2} - \frac{1}{4}$$

From GDC

$x = 1.46$

$f(1.46) = 8.47$

$y - y_1 = m(x - x_1)$

$\boxed{y - 8.47 = \frac{1}{4}(x - 1.46)}$

*Do **NOT** write solutions on this page.*

11. *[Maximum mark: 21]*

A random variable X has probability density function

$$f(x) = \begin{cases} ax+b, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}, \quad a, b \in \mathbb{R}.$$

(a) Show that $5a + 2b = 2$. [4]

Let $\mathrm{E}(X) = \mu$.

(b) (i) Show that $a = 12\mu - 30$.

(ii) Find a similar expression for b in terms of μ. [7]

Let the median of the distribution be 2.3.

(c) (i) Find the value of μ.

(ii) Find the value of the standard deviation of X. [10]

b ii) $38a + 15b = 6\mu$

$38(12\mu - 30) + 15b = 6\mu$

$456\mu - 1140 + 15b = 6\mu$

$15b = -450\mu + 1140$

$\boxed{b = 76 - 30\mu}$

a) $\int_2^3 (ax+b)\,dx = 1$

$\left.\frac{ax^2}{2} + bx\right|_2^3 = 1$

$\left(\frac{9a}{2} + 3b\right) - \left(\frac{4a}{2} + 2b\right) = 1$

$\frac{5a}{2} + b = 1$

$\boxed{5a + 2b = 2}$

$b = 1 - \frac{5}{2}a$

b i $E(x) = \int x f(x)\,dx = \mu$

$= \int_2^3 x(ax+b)\,dx = \mu$

$= \int_2^3 ax^2 + bx\,dx = \mu$

$= \left.\frac{ax^3}{3} + \frac{bx^2}{2}\right|_2^3 = \mu$

$\left(\frac{27a}{3} + \frac{9b}{2}\right) - \left(\frac{8a}{3} + 4b\right) = \mu$

$\left(\frac{19a}{3} + \frac{5}{2}b = \mu\right) 6$

$38a + 15b = 6\mu$

$38a = 6\mu - 15b$

$38a = 6\mu - 15\left(1 - \frac{5}{2}a\right)$

$38a = 6\mu - 15 + \frac{75}{2}a$

$\frac{1}{2}a = 6\mu - 15$

$\boxed{a = 12\mu - 30}$

11c. Let median = 2.3

11 ci) Find value of μ

$$\int_2^M (ax+b)\,dx = \frac{1}{2}$$

$$\int_2^{2.3} (ax+b)\,dx = \frac{1}{2}$$

$$\frac{ax^2}{2} + bx \Big|_2^{2.3} = \frac{1}{2}$$

$$0.645a + 0.3b = 0.5$$

$$.645(12\mu - 30) + 0.3(76 - 30\mu) = 0.5$$

$$\boxed{\mu = 2.34 \text{ or } \frac{295}{126}}$$

cii Find value of standard Deviation of X.

$$E(x) = 2.34 = \mu$$

$$\text{var}(x) = \int_2^3 x^2(ax+b)\,dx - (2.34)^2$$

$$= \int ax^3 + bx^2 \Big|^3$$

$$= \frac{ax^4}{4} + \frac{bx^3}{3} \Big|_2$$

$$= \left(\frac{81a}{2} + \frac{27b}{3}\right) - \left(\frac{16}{2}a + \frac{8}{3}b\right)$$

$$=$$

$$= \frac{65}{2}a + \frac{19}{3}b$$

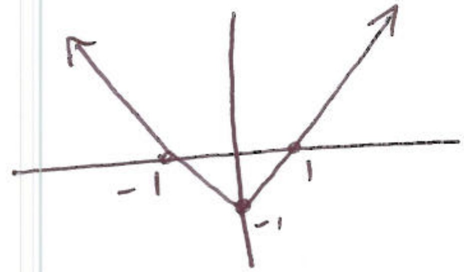
$$= \frac{65}{2}(12\mu - 30) + \frac{19}{3}(76 - 30\mu)$$

$$\text{var}(x) = 0.05813\ldots$$

$$\sigma = \sqrt{\text{var}(x)}$$

$$\boxed{\sigma = 0.241}$$

*Do **NOT** write solutions on this page.* → b i)



12. *[Maximum mark: 18]*

Let $f(x)=|x|-1$.

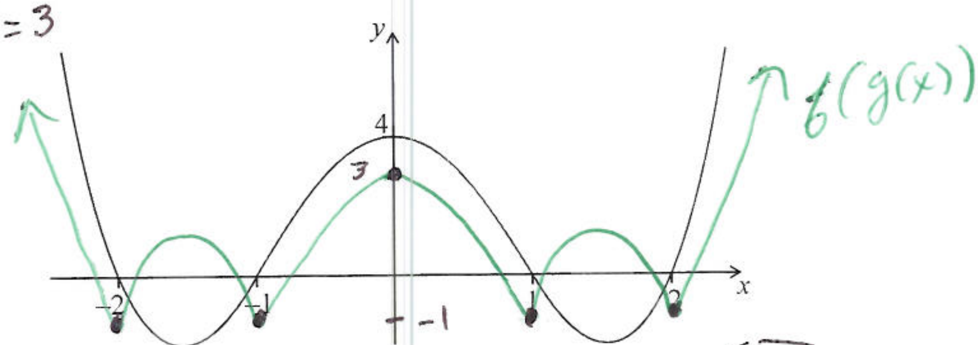
(a) The graph of $y=g(x)$ is drawn below.

iii) $f(g(0)) = f(4) = 3$

$f(g(2)) = f(0) = -1$

$f(g(-1)) = f(0) = -1$

$f(g(-2)) = f(0) = -1$



(i) Find the value of $(f\circ g)(1)$. $= f(g(1)) = f(0) = \boxed{-1}$

(ii) Find the value of $(f\circ g\circ g)(1)$ $= f(g(g(1))) = f(g(0)) = f(4) = \boxed{3}$

(iii) Sketch the graph of $y=(f\circ g)(x)$. Reflect lower part of graph across x-axis And y-int below 4. *[5]*

(b) (i) Sketch the graph of $y=f(x)$.

(ii) State the zeros of f. $\boxed{X = \pm 1}$ *[3]*

(c) (i) Sketch the graph of $y=(f\circ f)(x)$.

(ii) State the zeros of $f\circ f$ → $\boxed{X = 0, \pm 2}$ *[3]*

c i) $f(f(x)) = f(|x|-1) = ||x|-1|-1$

$= |x|-1-1$ or $-(|x|-1)-1$

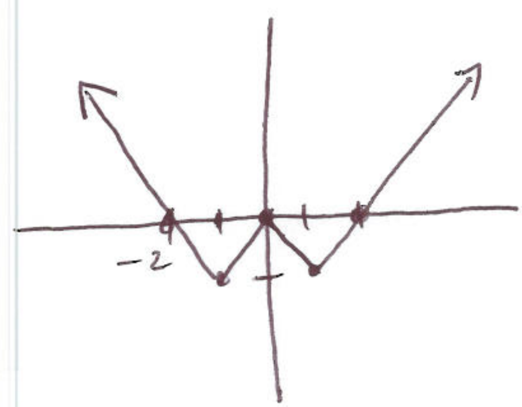
$= |x|-2$ or $-|x|+1-1$

$-|x|$

Zeros $|x| = 2$, $\boxed{X = \pm 2}$

Zeros $-|x| = 0$, $\boxed{X = 0}$

(This question continues on the following page)



$f(1) = -1$

$f(-1) = -1$

*Do **NOT** write solutions on this page.*

(Question 12 continued)

(d) Given that we can denote $\underbrace{f \circ f \circ f \circ \ldots \circ f}_{n \text{ times}}$ as f^n,

4 zeros ← (i) find the zeros of f^3; $x = \pm 1, \pm 3$

5 zeroes ← (ii) find the zeros of f^4; $x = 0, \pm 2, \pm 4$

9 zeroes ← (iii) deduce the zeros of f^8. $x = 0, \pm 2, \pm 4, \pm 6, \pm 8$ *[3]*

(e) The zeros of f^{2n} are $a_1, a_2, a_3, \ldots, a_N$.

(i) State the relation between n and N;

(ii) Find, and simplify, an expression for $\sum_{r=1}^{N} |a_r|$ in terms of n. *[4]*

e i) $(1, 3), (2, 5), (3, 7), \ldots$

$N = 2n + 1$

e ii) The formula for sum of arithmetic series

Method 1: $4(1 + 2 + 3 + \ldots + n) = \frac{4}{2} n(n+1)$

$\boxed{= 2n(n+1)}$

Method 2: $2(2 + 4 + 6 + \ldots + 2n) = \frac{2}{2} n(2n+2)$

$\boxed{= 2n(n+1)}$

$$\boxed{S_n = \frac{n}{2}(u_1 + u_n)}$$