Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Wahbeh
Isey

Tuesday 12 May 2015 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks].**

15 pages

2215–7203

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

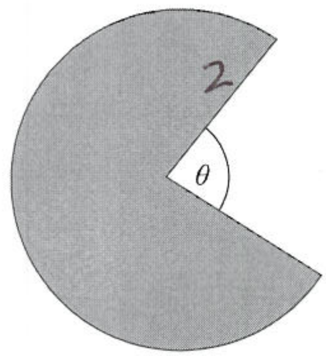
Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The logo, for a company that makes chocolate, is a sector of a circle of radius $2\,\text{cm}$, shown as shaded in the diagram. The area of the logo is $3\pi\,\text{cm}^2$.

diagram not to scale



(a) Find, in radians, the value of the angle θ, as indicated on the diagram. [3]

(b) Find the total length of the perimeter of the logo. [2]

a) $A = \pi r^2 \Rightarrow A = \pi(2)^2 \Rightarrow A = 4\pi$

$\therefore$ Area of non shaded sector $= 4\pi - 3\pi = \pi$

$A = \frac{1}{2}\theta r^2$ area of sector (see formula)

$\pi = \frac{1}{2}\theta 2^2$

$\pi = 2\theta$

$\boxed{\therefore \theta = \frac{\pi}{2}}$

b) $L = \theta r$ length of arc see formula

$L = \frac{\pi}{2}(2)$

$L = \pi$ (of non shaded)

$C = 2\pi r$

$C = 2\pi(2)$

$C = 4\pi$

$\therefore$ length of shaded arc $= 4\pi - \pi = 3\pi$

$\boxed{P = 3\pi + 4}$

2. [Maximum mark: 5]

A mathematics test is given to a class of 20 students. One student scores 0, but all the other students score 10.

(a) Find the mean score for the class. [2]

(b) Write down the median score. [1]

(c) Write down the number of students who scored

(i) above the mean score;

(ii) below the median score. [2]

a) $\frac{19(10)+0}{20} = \frac{190}{20} = \frac{19}{2} = 9.5 = \bar{x}$

b) median $= 10$

c) i 19

ii 1

3. [Maximum mark: 5]

(a) Find $\int(1+\tan^2 x)\,\mathrm{d}x$. [2]

(b) Find $\int\sin^2 x\,\mathrm{d}x$. [3]

a) $\int(1+\tan^2 x)dx$　　$\sec^2 x = 1 + \tan^2 x$

$\therefore \int \sec^2 x \, dx = \boxed{\tan x + c}$

b) $\int \sin^2 x \, dx$　　$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\therefore \int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$

$\boxed{= \frac{1}{2}x - \frac{1}{4}\sin 2x + c}$

4. [Maximum mark: 5]

(a) Expand $(x+h)^3$. [2]

(b) Hence find the derivative of $f(x) = x^3$ from first principles. [3]

1
1 1
1 2 1
1 3 3 1

a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

b) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$

$\boxed{f'(x) = 3x^2}$

5. [Maximum mark: 6]

The functions f and g are defined by $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$ and $g(x) = p\sin x + qx + r$, $x \in \mathbb{R}$ where a, b, c, p, q, r are real constants.

(a) Given that f is an even function, show that $b = 0$. [2]

(b) Given that g is an odd function, find the value of r. [2]

The function h is both odd and even, with domain $\mathbb{R}$.

(c) Find $h(x)$. [2]

a) $f(x) = f(-x)$ even function

$f(-x) = a(-x)^2 + b(-x) + c$

$f(-x) = ax^2 - bx + c$

$ax^2 - bx + c = ax^2 + bx + c$

$-ax^2 + bx - c \quad -ax^2 + bx - c$

$2bx = 0$

$\boxed{\therefore b = 0}$

b) $g(-x) = -g(x)$ odd function.

$p\sin(-x) + q(-x) + r = -p\sin x - qx - r$

Note: $\sin(-x) = -\sin x$ in Q IV

$-p\sin x - qx + r = -p\sin x - qx - r$

$2r = 0$

$\boxed{\therefore r = 0}$

c) $h(x) = h(-x) = -h(x) \Rightarrow h(-x) = -h(x)$

$\therefore 2h(x) = h(-x)$

$\therefore 2h(x) = -h(x)$

$3h(x) = 0$

$\boxed{h(x) = 0}$

6. [Maximum mark: 7]

A function f is defined by $f(x)=\dfrac{3x-2}{2x-1}$, $x\in\mathbb{R}$, $x\neq\dfrac{1}{2}$.

(a) Find an expression for $f^{-1}(x)$. [4]

(b) Given that $f(x)$ can be written in the form $f(x)=A+\dfrac{B}{2x-1}$, find the values of the constants A and B. [2]

(c) Hence, write down $\int\dfrac{3x-2}{2x-1}\,\mathrm{d}x$. [1]

a) interchange x for y and solve for y.

$x=\dfrac{3y-2}{2y-1}$

$x(2y-1)=3y-2$

$2xy-x=3y-2$

$2xy-3y=x-2$

$y(2x-3)=x-2$

$y=\dfrac{x-2}{2x-3}$

$\boxed{\therefore f^{-1}(x)=\dfrac{x-2}{2x-3}}$

b) Method 2

$\dfrac{3x-2}{2x-1}=A+\dfrac{B}{2x-1}$

$\dfrac{3x-2}{2x-1}=\dfrac{A(2x-1)+B}{2x-1}$

$\therefore 3x-2=2Ax-A+B$

Equating coefficients of x.

$3=2A$

$\boxed{A=3/2}$

$-2=-A+B$ Equating

$-2=-\frac{3}{2}+B \rightarrow \boxed{B=-1/2}$

b) Method 1

$$\begin{array}{r|l} & \tfrac{3}{2} \\ 2x-1 & 3x-2 \\ & -(3x-3/2) \\ \hline & -1/2 \end{array}$$

$\therefore \dfrac{3}{2}-\dfrac{1/2}{2x-1}$

$\boxed{\therefore A=3/2,\ B=-1/2}$

c) $\int\dfrac{3x-2}{2x-1}\,dx=\int\left(\dfrac{3}{2}+\dfrac{-1/2}{2x-1}\right)dx$

$\boxed{=\dfrac{3}{2}x-\dfrac{1}{4}\ln|2x-1|+C}$

7. [Maximum mark: 5]

Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36$, $x \in \mathbb{R}$.

Look @ pages 136-136 HL book. 3.4

(a) For the polynomial equation $p(x) = 0$, state

(i) the sum of the roots; Formula $= \dfrac{-a_{n-1}}{a_n}$

(ii) the product of the roots. [3] Formula $= \dfrac{(-1)^n a_0}{a_n}$

A new polynomial is defined by $q(x) = p(x+4)$.

highest degree

(b) Find the sum of the roots of the equation $q(x) = 0$. [2]

a i) $\sum_{r=1}^{5} x_r = \dfrac{-a_4}{a_5} = \boxed{\dfrac{-1}{2}}$

product

ii $\prod_{r=1}^{5} x_r = \dfrac{(-1)^n a_0}{a_n} = \dfrac{(-1)^5\, 36}{2} = \boxed{-18}$

b) $q(x) = 2(x+4)^5 + (x+4)^4 - 13(x+4)^2 + 72(x+4) + 36$

Look for highest 2 powers!

$2(x+5)^5 = 2\left[\binom{5}{0}x^5 + \binom{5}{1}x^4(4) + \binom{5}{2}x^3(4)^2 + \dots\right]$

$= 2[1x^5 + 20x^4 + \dots] = 2x^5 + 40x^4 + \dots$

$(x+4)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3(4) + \binom{4}{2}x^2(4)^2 + \dots$

$= x^4 + 16x^3 + \dots$

$2(x+5)^5 + (x+4)^4 = 2x^5 + 40x^4 + \dots + x^4$

$= 2x^5 + 41x^4 + \dots$

$\therefore$ sum of roots: $\sum_{r=1}^{5} r_1 = \dfrac{-a_4}{a_5} = \boxed{-\dfrac{41}{2} = -20.5}$

8. [Maximum mark: 7]

By using the substitution $u = e^x + 3$, find $\int \frac{e^x}{e^{2x} + 6e^x + 13} dx$.

$u = e^x + 3 \longrightarrow e^x = u - 3$
$du = e^x dx$

$e^{2x} + 6e^x + 13 = (e^x)^2 + 6e^x + 13$
$= (u-3)^2 + 6(u-3) + 13$

$\int \frac{e^x}{(e^x)^2 + 6e^x + 13} dx = \int \frac{1}{(u-3)^2 + 6(u-3) + 13} du$

$= \int \frac{1}{u^2 - 6u + 9 + 6u - 18 + 13} du$

$= \int \frac{1}{u^2 + 4} du$

Look @ IB formula

OR

$\int \frac{1}{u^2+4} du$

$\int \frac{1}{4(\frac{1}{4}u^2 + 1)} du$

$\frac{1}{4} \int \frac{1}{(\frac{1}{2}u)^2 + 1} du$

$\frac{1}{4} \frac{\arctan(\frac{1}{2}u)}{1/2} + c$

$\frac{1}{2} \arctan \frac{1}{2}u + c$

$\boxed{\frac{1}{2} \arctan\left(\frac{e^x+3}{2}\right) + c}$

$= \frac{1}{2} \arctan\left(\frac{u}{2}\right)$

$\boxed{= \frac{1}{2} \arctan\left(\frac{e^x+3}{2}\right) + c}$

Note: $y = \tan^{-1} \heartsuit$
$y' = \frac{1}{1+\heartsuit^2} \cdot \heartsuit'$

9. [Maximum mark: 9]

The functions f and g are defined by $f(x) = 2x + \frac{\pi}{5}$, $x \in \mathbb{R}$ and $g(x) = 3\sin x + 4$, $x \in \mathbb{R}$.

(a) Show that $g \circ f(x) = 3\sin\left(2x + \frac{\pi}{5}\right) + 4$. [1]

(b) Find the range of $g \circ f$. [2]

(c) Given that $g \circ f\left(\frac{3\pi}{20}\right) = 7$, find the next value of x, greater than $\frac{3\pi}{20}$, for which $g \circ f(x) = 7$. [2]

(d) The graph of $y = g \circ f(x)$ can be obtained by applying four transformations to the graph of $y = \sin x$. State what the four transformations represent geometrically and give the order in which they are applied. [4]

a) $g \circ f(x) = g(f(x)) = g\left(2x + \frac{\pi}{5}\right) = \boxed{3\sin\left(2x + \frac{\pi}{5}\right) + 4}$

b) Range of f = all Real
Range of $g = [1, 7]$ $\sin x = -1$ … 1
What do they both have in common? $[1, 7]$
$\therefore$ range of $g \circ f = [1, 7]$

c) $g \circ f\left(\frac{3\pi}{20}\right) = 3\sin\left[2x + \frac{\pi}{5}\right] + 4 = 7$
$= \sin\left(2x + \frac{\pi}{5}\right) = 1$ $(0, 1)$

$2x + \frac{\pi}{5} = \frac{\pi}{2} + 2\pi n$

$2x = \frac{\pi}{2} - \frac{\pi}{5} + 2\pi n$

$x = \left(\frac{3\pi}{10} + 2\pi n\right)\frac{1}{2}$

$n = 0 \Rightarrow x = \frac{3\pi}{20}$

$n = 1$

$x = \frac{3\pi}{20} + \pi$

$\boxed{x = \frac{23\pi}{20}}$

(This question continues on the following page)

(Question 9 continued)

$g \circ f(x) = 3 \sin\left(2x + \frac{\pi}{5}\right) + 4$

$= 3 \sin\left[2\left(x + \frac{\pi}{10}\right)\right] + 4$

which has this form: $a \sin[b(x+c)] + d$

1) Horizontal stretch by a factor of $\frac{1}{2}$ $\rightarrow b$

2) Horizontal shift left $\frac{\pi}{10}$ units $\longrightarrow c$

3) Vertical stretch by a factor of 3 $\longrightarrow a$

4) Vertical shift up 4 units $\longrightarrow d$

Note: order does not matter as long as stretch comes before shift.

10. [Maximum mark: 6]

A football team, Melchester Rovers are playing a tournament of five matches.

The probabilities that they win, draw or lose a match are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively.

These probabilities remain constant; the result of a match is independent of the results of other matches. At the end of the tournament their coach Roy loses his job if they lose three **consecutive** matches, otherwise he does not lose his job. Find the probability that Roy loses his job.

Roy loses his job if:

A – First 3 games are lost (last 2 don't matter)
LLL

B – First game not lost, 3 middle games lost, last game don't matter.
L' LLL

C – First game any result, second game not lost, last 3 lost.
L' LLL

$P(A) = \left(\frac{1}{3}\right)^3$

$P(B) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$

$P(C) = 1 \times \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$

since events are mutually exclusive

Total probability $= \left(\frac{1}{3}\right)^3 + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{2}{3}\left(\frac{1}{3}\right)^3$

$= \frac{1}{27} + \frac{2}{81} + \frac{2}{81}$

$= \frac{3}{81} + \frac{4}{81} = \boxed{\frac{7}{81}}$

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Let $y(x) = xe^{3x}, x \in \mathbb{R}$.

(a) Find $\frac{dy}{dx}$. $= 1e^{3x} + xe^{3x}(3) = \boxed{e^{3x} + 3xe^{3x}}$ [2]

(b) Prove by induction that $\frac{d^n y}{dx^n} = n3^{n-1}e^{3x} + x3^n e^{3x}$ for $n \in \mathbb{Z}^+$. [7]

(c) Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. Justify whether any such point is a maximum or a minimum. [5]

(d) Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. [5]

(e) Hence sketch the graph of $y(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [2]

b) Let $P(n)$ be the statement $\frac{d^n y}{dx^n} = n3^{n-1}e^{3x} + x3^n e^{3x}$

Prove $n=1$

LHS of $P(1)$ is $\frac{dy}{dx} = \boxed{e^{3x} + 3xe^{3x}}$ Same

RHS of $P(1)$ is $\frac{d^1 y}{dx^1} = 1(3^{1-1})e^{3x} + x(3^1)e^{3x} = \boxed{e^{3x} + 3xe^{3x}}$

Since LHS = RHS, $\boxed{P(1) \text{ is true.}}$

Assume $P(K)$ is true and attempt to prove $P(K+1)$ is true.

Assuming $\frac{d^K y}{dx^K} = K3^{K-1}e^{3x} + x3^K e^{3x}$

So $\frac{d^{K+1} y}{dx^{K+1}} = \boxed{(K+1)3^K e^{3x} + x3^{K+1}e^{3x}}$ Same

$\frac{d^{K+1} y}{dx^{K+1}} = \frac{d}{dx}\left(\frac{d^K y}{dx^K}\right) = K3^{K-1}e^{3x}(3) + 1(3^K e^{3x}) + x3^K e^{3x}(3)$

$= K3^{K-1+1}e^{3x} + 3^K e^{3x} + x3^{K+1}e^{3x}$

$\boxed{= 3^K e^{3x}(K+1) + x3^{K+1}e^{3x}}$

Continue

11 b continue

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true.
Then $P(n)$ is true.

c) $e^{3x} + 3x e^{3x} = 0$ set $\frac{dy}{dx} = 0$ to find Max/Min

$e^{3x}(1+3x) = 0$

$e^{3x} = 0$ or $1+3x=0$

$x \neq 0$, $\boxed{x = -\frac{1}{3} \text{ C.V.}}$

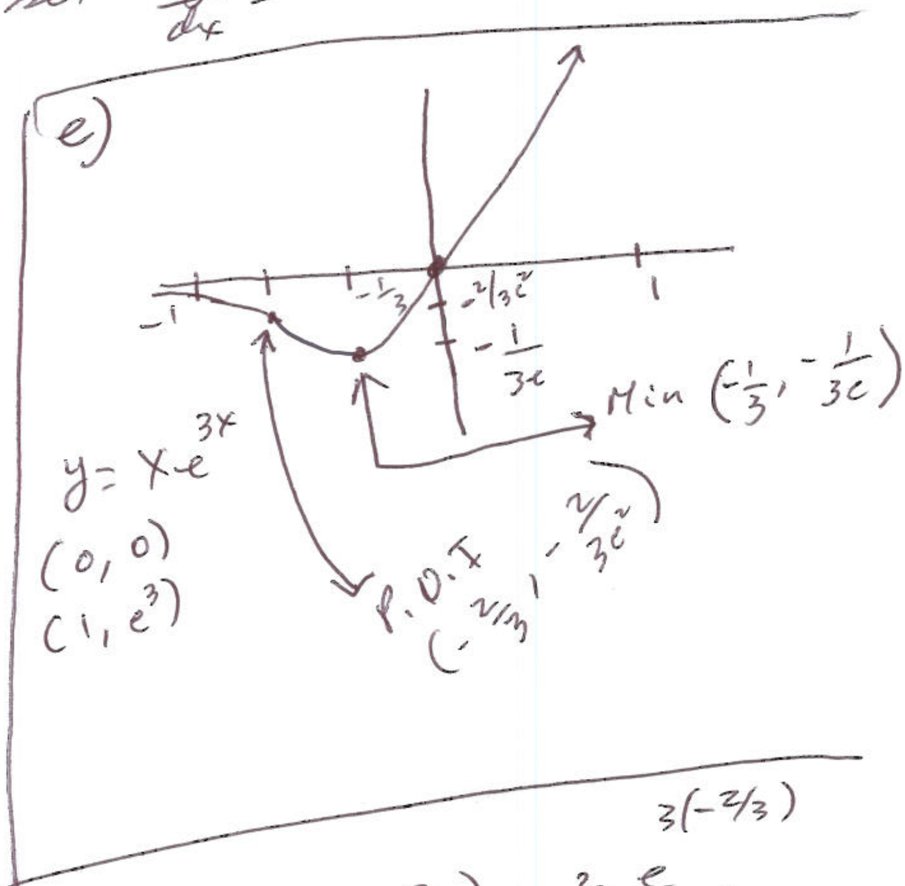
y': $-$ | $+$ at $-\frac{1}{3}$

$\boxed{\therefore x = -\frac{1}{3} \text{ is Min.}}$

$f(-\frac{1}{3}) = -\frac{1}{3} e^{3(-1/3)}$

$= -\frac{1}{3} e^{-1}$

$\boxed{f(-1/3) = \frac{-1}{3e}}$



d) $\frac{d^2y}{dx^2} = 3e^{3x} + 3e^{3x} + 9x e^{3x}$

$= 6e^{3x} + 9x e^{3x}$

$0 = 3e^{3x}(2+3x)$

$3e^{3x} = 0$ or $2+3x=0$

$x \neq 0$ $\boxed{x = -\frac{2}{3} \text{ C.V.}}$

$f(-\frac{2}{3}) = -\frac{2}{3} e^{3(-2/3)}$

$= -\frac{2}{3} e^{-2}$

$\boxed{f(-\frac{2}{3}) = \frac{-2}{3e^2}}$

y: $\cap$ $\cup$

y'': $-$ | $+$ at $-2/3$

$x = -\frac{2}{3}$ point of inflection since $y''=0$ and there is a sign change (concave down to concave up).

Do **not** write solutions on this page.

12. [Maximum mark: 18]

Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 0$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

(a) (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant.

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d and n. [4]

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

(b) (i) Find S_n, in terms of a, d and n.

(ii) Find the values of d for which $\sum_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d.

(iv) Given that $S_\infty = 2^{a+1}$ find the value of d. [8]

Let $\{w_n\}$, $n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$.

(c) Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q. [6]

a i) $u_n = a + (n-1)d$ (see formula for A.S.)

$v_n = 2^{a+(n-1)d}$

$v_{n+1} = 2^{a+(n-1+1)d}$

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}} = \frac{2^a \times 2^{nd}}{2^a \times 2^{nd} \times 2^{-d}} = \frac{1}{2^{-d}} = \boxed{2^d \text{ a constant.}}$

12 aii) $u_n = u_1 r^{n-1}$ (General Geometric Seq.)

$v_n = 2^{a+(n-1)d}$

$v_n = 2^a (2^{(n-1)d})$ $\therefore$ First term $= 2^a$

aiii) $v_n = 2^a (2^d)^{n-1}$ Common ratio $= 2^d$

Let S_n be the sum of the first n terms of seq. $\{v_n\}$.

bi) Find S_n. $S_n = \frac{u_1 (r^n - 1)}{r-1}$ (See IB Formula for finite)

$S_n = \frac{2^a((2^d)^n - 1)}{2^d - 1}$, note $u_1 = 2^a$, $r = 2^d$

bii) $\sum_{i=1}^{\infty} v_i$ exists. Find values of d.

To exist, a geometric sequence $|r| < 1$

$\therefore -1 < 2^d < 1$

$-1 < 2^d$ or $2^d < 1$

$\log(-1) < \log 2^d$ $\quad$ $\log 2^d < \log 1$

$\log(-1) < d \log 2$ $\quad$ $d \log 2 < 0$

$d =$ DNE $\quad$ $\boxed{d < 0}$

12) You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

b iii) Write down S_∞. $S_n = \frac{u_1}{1-r}$ (see IB Formula for infinite seq)

$$S_\infty = \frac{2^a}{1-2^d}$$

iv) $S_\infty = 2^{a+1}$ is given. Find d. (sub in for iii)

$$2^{a+1} = \frac{2^a}{1-2^d}$$

$$1-2^d = \frac{2^a}{2^{a+1}}$$

$$1-2^d = \frac{1}{2}$$

$$2^d = \frac{1}{2}$$

$$\log 2^d = \log \frac{1}{2}$$

$$d \log 2 = \log 1 - \log 2$$

$$d = \frac{\log 1 - \log 2}{\log 2}$$

$$d = \frac{0 - \log 2}{\log 2}$$

$$\boxed{d = -1}$$

c) Given $z_n = \ln w_n$

Find $\sum_{i=1}^{n} z_i$. Give answer in the form $\ln k$, where k is in terms of n, p, q.

$w_n = p q^{n-1}$ (General formula for G.S.)

$$z_n = \ln(pq^{n-1})$$

$$z_n = \ln p + \ln q^{n-1}$$

$$z_n = \ln p + (n-1)\ln q$$

$$z_n = u_1 + (n-1)d$$

has this form ∴ it is an Arith. Seq. w/ $u_1 = \ln p$, $d = \ln q$.

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$= \frac{n}{2}\left(2\ln p + (n-1)\ln q\right)$$

$$= n\left(\ln p + \frac{(n-1)}{2}\ln q\right)$$

$$= n\left(\ln p + \ln q^{\frac{n-1}{2}}\right)$$

$$= n\left(\ln p q^{\frac{n-1}{2}}\right)$$

$$\boxed{S_n = \ln p^n q^{\frac{n(n-1)}{2}}}$$

Do **not** write solutions on this page.

13. [Maximum mark: 21]

Two lines l_1 and l_2 are given respectively by the equations $\boldsymbol{r}_1 = \overrightarrow{\mathrm{OA}} + \lambda \boldsymbol{v}$ and $\boldsymbol{r}_2 = \overrightarrow{\mathrm{OB}} + \mu \boldsymbol{w}$ where $\overrightarrow{\mathrm{OA}} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{\mathrm{OB}} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and O is the origin. Let P be a point on l_1 and let Q be a point on l_2.

(a) Find $\overrightarrow{\mathrm{PQ}}$, in terms of λ and μ. [2]

(b) Find the value of λ and the value of μ for which $\overrightarrow{\mathrm{PQ}}$ is perpendicular to the direction vectors of both l_1 and l_2. [5]

(c) Hence find the shortest distance between l_1 and l_2. [3]

(d) Find the Cartesian equation of the plane Π, which contains line l_1 and is parallel to the direction vector of line l_2. [5]

Let $\overrightarrow{\mathrm{OT}} = \overrightarrow{\mathrm{OB}} + \eta\,(\boldsymbol{v} \times \boldsymbol{w})$.

(e) Find the value of η for which the point T lies in the plane Π. [2]

(f) For this value of η, calculate $\left|\overrightarrow{\mathrm{BT}}\right|$. [2]

(g) State what you notice about your answers to (c) and (f), and give a geometrical interpretation of this result. [2]

a) $r_1 = (i + 2j + 3k) + (i + j + k)\lambda \Rightarrow OP$

$r_2 = (2i + j - k) + (i - j + 2k)\mu \Rightarrow OQ$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}1\\-1\\-4\end{pmatrix} + \begin{pmatrix}1\\-1\\2\end{pmatrix}\mu - \begin{pmatrix}1\\1\\1\end{pmatrix}\lambda$

$\overrightarrow{PQ} = (1 + \mu - \lambda)i + (-1 - \mu - \lambda)j + (-4 + 2\mu - \lambda)k.$

13b)

Use of scalar product $\perp$ to $i+j+k$ gives.

$(1+\mu-\lambda)i + (-1-\mu-\lambda)j + (-4+2\mu-\lambda)k = 0$

$-4+2\mu-3\lambda=0 \Rightarrow \boxed{2\mu-3\lambda=4}$

Use of scalar product $\perp$ to $i-j+2k$ gives

$(1+\mu-\lambda)i - (-1-\mu-\lambda)j + 2(-4+2\mu-\lambda) = 0$

$-6+6\mu-2\lambda=0 \Rightarrow \boxed{6\mu-2\lambda=6}$

$3(2\mu-3\lambda=4) \Rightarrow \quad 6\mu-9\lambda=12$

$6\mu-2\lambda=6 \quad\quad -(6\mu-2\lambda=6)$

$-7\lambda=6 \Rightarrow \boxed{\lambda=-\frac{6}{7}}$

$\therefore\ 2\mu-3\left(-\frac{6}{7}\right)=4$

$2\mu+\frac{18}{7}=4$

$2\mu=\frac{28}{7}-\frac{18}{7}$

$2\mu=\frac{10}{7}$

$\boxed{\mu=\frac{10}{14}=\frac{5}{7}}$

c) Find shortest distance bt. L_1 and L_2

Note: Since $\vec{PQ} \perp$ to L_1 and L_2 this makes $\vec{PQ}$ shortest distance.

Sub λ and μ for $\vec{PQ}$

$\vec{PQ} = \left(1+\frac{5}{7}+\frac{6}{7}\right)i + \left(-1-\frac{5}{7}+\frac{6}{7}\right)j + \left(-4+2\times\frac{5}{7}+\frac{6}{7}\right)k$

$-\frac{28}{7}+\frac{10}{7}+\frac{6}{7}$

$\vec{PQ} = \frac{18}{7}i - \frac{6}{7}j - \frac{12}{7}k$

$\vec{PQ} = \frac{6}{7}\begin{pmatrix}3\\-1\\-2\end{pmatrix} \Rightarrow |\vec{PQ}| = \frac{6}{7}\sqrt{3^2+1^2+2^2} = \boxed{\frac{6}{7}\sqrt{14}}$

$9+1+4$

13 d). Find cartesian eq. of plane Π, which contains L_1, and is // to direction of L_2.

$ax + by + cz + d = 0$ General eq. of plane where normal $= \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

To find normal use cross product of direction of L_1 and L_2

direction of $L_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, direction of $L_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$L_1 \times L_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{matrix} (2+1)i \\ -(2-1)j \\ (-1-1)k \end{matrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = n$$

$$3x - 1y - 2z = \underset{3-2-6}{3(1) - 1(2) - 2(3)} = 0$$

where $(1, 2, 3)$ is a point on L_1.

$$\boxed{3x - y - 2z + 5 = 0}$$

let $\overrightarrow{OT} = \overrightarrow{OB} + \eta(v \times w)$

e) Find value of η for which point T lies in plane Π.

$$OT = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \eta\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$T = (2 + 3\eta, 1 - \eta, -1 - 2\eta)$

Plug T in plane

$$3(2+3n) - (1-n) - 2(-1-2n) + 5 = 0$$

$$6 + 9n - 1 + n + 2 + 4n + 5 = 0$$

$$12 + 14n = 0 \Rightarrow \boxed{\eta = -\frac{6}{7}}$$

13 b) for $n = -\frac{6}{7}$ calculate $|\overrightarrow{BT}|$.

$$\overline{OT} = \begin{pmatrix} 2+3\times-\frac{6}{7} \\ 1--\frac{6}{7} \\ -1-2(-\frac{6}{7}) \end{pmatrix} = \begin{pmatrix} \frac{14}{7}-\frac{18}{7} \\ \frac{7}{7}+\frac{6}{7} \\ -\frac{7}{7}+\frac{12}{7} \end{pmatrix} = \begin{pmatrix} -\frac{4}{7} \\ \frac{13}{7} \\ \frac{5}{7} \end{pmatrix}$$

$$\overrightarrow{BT} = OT - OB = \begin{pmatrix} -\frac{4}{7} \\ \frac{13}{7} \\ \frac{5}{7} \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{18}{7} \\ \frac{6}{7} \\ \frac{12}{7} \end{pmatrix} = \frac{6}{7}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{BT}| = \frac{6}{7}\sqrt{3^2+1^2+2} = \boxed{\frac{6}{7}\sqrt{14}}$$

g) from part c $|\overrightarrow{PQ}| = \frac{6}{7}\sqrt{14}$
from part b $|\overrightarrow{BT}| = \frac{6}{7}\sqrt{14}$ Same

$\overrightarrow{BT}$ is $\perp$ to both Π and L_2 so its length is shortest distance between Π and L_2, which is shortest distance between L_1 and L_2.